Optimal servo system design for linear distributed parameter systems with unknown input disturbances

Kenji Shirai[1] Yoshinori Amano[2]

Abstract

We propose an optimal servo system design for linear distributed parameter systems with unknown disturbance inputs in this study. An equivalent distribution disturbance method is used to estimate the unknown input. In an infinite dimensional system, it is difficult to determine the characteristics of the disturbance. Therefore, we propose an optimal input system configuration that limits the spatial distribution to a local region. Generally, the use of the equivalent distribution disturbance is a valuable method for estimating the input signal for such a system.

Keyword: equivalent distribution disturbance, linear distributed parameter systems, Generalized random variable, optimal control.

1 Introduction

With respect to mathematical modeling for a linear distributed parameter system, we have reported that by creating a state in which the production density of each process corresponds to physical propagation, the manufacturing process is most appropriately described using a diffusion equation [1, 2]. In other words, if the potential of the production field (stochastic field) is minimized, the equation is defined by the production density function $S_i(t,x)$ and the constraint is described using an advective diffusion equation to determine the transportation speed ρ [2]. On the other hand, with respect to a bilinear partial differential equation (BPDE), Dr.Shima proposed an optimal control system for heat exchanging system. The mathematical model is described by BPDE [3].

In our previous study, we proposed a mathematical model for a thermal reaction process of external heating equipment. The new control system design for this process, which treats a heat source flowing model for an externally attached device is proposed. The equation of a distributed parameter system as a coupled system with the heat reaction process is presented [4, 5]. We also proposed that the target control system can be configured using the control parameter of the overall heat exchange coefficient (OHEC), which is given using a linear approximation from BPDE to an ordinary differential equation (ODE). Generally, in case of a physical variable in the mathematical model, there are two forms that depend on only a time and a time/space . In only time dependency, the mathematical model is described by an ordinary differential equation. In the time/space dependency, it is described by a partial differential equation. Furthermore, it is sometimes described by a stochastic differential equation by considering a disturbance (noise) to the system from outside. We have reported on a production process represented by a partial differential equation (including stochastic systems) in our previous research[2, 5, 6, 7].

In recent years, there has been made several reports in the lumped parameter system to improve the control performance of the control system. These proposed methods impose rank conditions on unknown input disturbances or require a maximum value of disturbances[8]. Among them, a method that has attracted attention is a disturbance estimation method that defines an equivalent input disturbance and uses the output of the controlled target[9].

On the other hand, there are many irregular input disturbances in the distributed parameter system that have distributions in both time and space. In constructing a mathematical model for such irregular disturbances, the first requirement is an exact mathematical description of the noise distributed in space. There is a direct method as a theoretical method for the estimation or control problem of a distributed parameter system subjected to random disturbance. In this method, it is necessary to introduce the concept of a random field with time and space variables as parameters. In the direct method, it is necessary

[1]Information and Culture Faculty

[2]Kyohnan Elecs co.,LTD

to clarify the relation between the theoretical and real aspects of the system state. With respect to the mathematical model of a noise, A method of applying white noise extracted from a finite or an infinite number of noise sources to the system was introduced[10]. Using the Wiener process for white noise is a mathematically exact expression, but there remains a problem in handling an independent Wiener process for each point in the space[11].

Another way of thinking is to use a functional space analysis. This method derives the noise applied to the system as a stochastic integral in the functional space, and has the advantage of a strict theoretical expansion However, the physical meaning of the development equation obtained as a mathematical model is not clear when using a partial differential equation directly. It should be noted that Dr. Sunahara has a great deal of achievement in clarifying the physical background of the Wiener process and Wiener integration[12]. Then, Dr.Sunahara introduced a Generalized random variable theory to handle the independent Wiener process for each point in the space[12].

In this study, we report the configuration of a zero-balance regulator system or a servo system with a disturbance against a linear distributed parameter system. First, it is difficult to determine the characteristics of a disturbance in an infinite dimensional system, and for such a system, it is significant to estimate an equivalent disturbance, that is, an input signal. Dr. Yoh et al, has reported that they are effective against ordinary differential equation system models[9]. Finally, we present the system configuration and optimal input for distributed parameter systems with disturbances.

2 Target system description

2.1 Diffusion model for linear distributed parameter systems

First, it is difficult to determine the characteristics of disturbance in an infinite dimensional system. Therefore, it is significant to estimate the input signal as an equivalent disturbance. Dr. Yoh et al, has reported that they are effective against ordinary differential equation system models[9].

A step disturbance is applied as input disturbance in this paper. We define the mathematical model of the state variable $C(t,x)$ to be controlled as follows.

Definition 2.1 *Mathematical model of the state variable $C(t,x)$*

$$\frac{\partial C(t,x)}{\partial t} = A_x C(t,x) + B_x f(t) + B_{dx}\xi(t) \tag{2.1}$$

where, the variables used in equation (2.1) are as follows.

A_x is originally a function related to time and space, but it is limited to local disturbances in this paper. Therefore, A_x is discussed as a function of time. B_x and B_{dx} denote a spatial distribution function and a spatial distribution function for disturbances respectively. $f(t)$ and $\xi(t)$ denote a control function and a step disturbance respectively.

Definition 2.2 *From Figure 1, when the state variable $\lim_{(t,x)\to\pm 0}\{C(t,x)=0, f(t)=0\}$ or $\lim_{(t,x)\to\pm\infty}\{C(t,x)=0, f(t)=0\}$. The output of the control target $C(t,x)$ for the disturbance $\xi(t)$ is $y(t,x)$, and the output of the control target $\bar{C}(t,x)$ for the disturbance $\xi(t)$ is $\bar{y}(t,x)$. If $\bar{y}(t,x)=y(t,x)$ for all (t, x), $\bar{\xi}(t)$ is called an equivalent input disturbance of $\xi(t)$ in Figure 2.*

According to the definition and Figure **??**, we present the control target model with an equivalent input disturbance as follows:

$$\begin{aligned}\frac{\partial \bar{C}(t,x)}{\partial t} &= A_x\bar{C}(t,x) + b_x f(t) + b_x\bar{\xi}(t)\\ \bar{y}(t,x) &= \boldsymbol{C}\{\bar{C}(t,x)\}\end{aligned} \tag{2.2}$$

where b_x denotes the equivalent distribution of B_d and B_{dx}

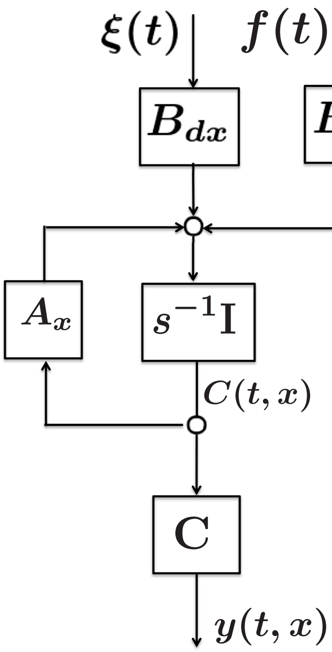

Fig. 1: Control target model

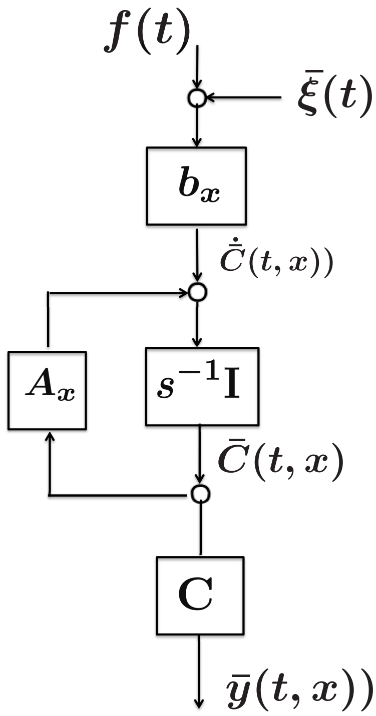

Fig. 2: Control target model with an equivalent input disturbance

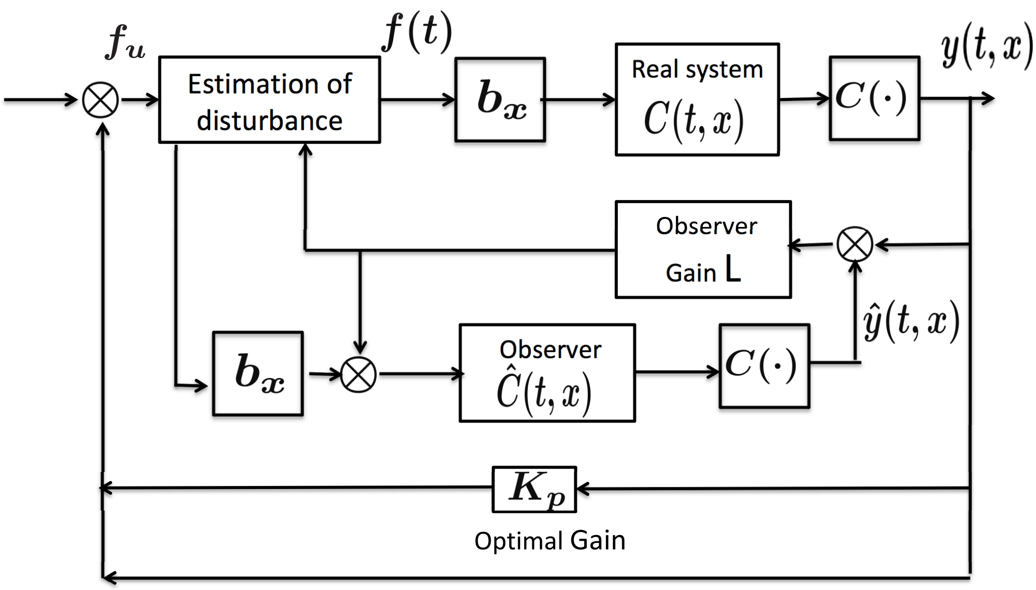

Fig. 3: System configulation

2.2 Spatial distribution disturbance

According to Dr.Sunahara's paper, we define the spatial distribution disturbance under Winner process as follows[12]:

Definition 2.3 *It is defined in n-dimensional Euclidean space and depends on the parameter t. When random field B has the following characteristics, it is called a distributed Winner process.*

1. *For each $t \in T$, $B_{t,x}$ has a pedestal within $T \times D$ and there is $\sup[\varphi] \cap \sup[_{\pi-h}\varphi] \subset D$ that becomes $\varphi \in K(R^n)$ and $h \in R^n$. The following equation holds for both of h and an arbitrary real number λ, and D represents a bounded area.*

$$P[B_{t,x}(\varphi) \leq \lambda] = P[\tau_h B_{t,x}(\varphi) \leq \lambda]$$

2. *The value $B_t(\varphi)(\varphi \in K(R^n))$ of a Generalized random variable $B_{t,x}$ and $\sup[\varphi] \not\subseteq R^n - D$ represent a Winner process, and the increment is zero on average and the variance is $|t-s| \int_{R^n} \varphi^2(x)dx$.*

3. *For each $t \in T$, $B_{t,x}$ is a random field with an independent value at each point in R^n and zero rank.*

Theorem 2.1 $\frac{\partial B_{t,x}}{\partial t}$ *is a spatial disturbance white disturbance.*

From Definition 2.3, the singularity of the distributed white disturbance with respect to the time variable t has been eliminated, but the singularity with respect to the spatial variable x has not yet been eliminated. To solve this, the following function is used to approximate a differentiable function ρ^ε which is the Friedrichs's mollifier[12].

Definition 2.4 *The regularization related to ρ^ε of the distributed Wiener process $B_{t,x}$ is called a regularized distributed Wiener process and is represented by $B_t^\varepsilon(x)$. Here, ρ^ε is the Friedrichs's mollifier operator, which presents the Appendix A as an example. That is, the following equation is satisfied.*

$$B_t^\varepsilon \in L_p(D) \cap K(R^n), \quad t \in T, \quad p \geq 1$$

Theorem 2.2 *B_t^ε converges to $B_{t,x}$ in the sense of a Generalized random variable when $\varepsilon \to 0$.*

Please refer to the literature of the above for certification[12, 7].

2.3 Diffusion type distributed parameter system model using spatial disturbance

From Subsection 2.2, the diffusion type distributed parameter system model using spatial disturbance is as follows:

$$\frac{\partial \bar{C}(t,x)}{\partial t} = A_x \bar{C}(t,x) + \varphi(x,x^{'})f(t) + d(t,x)\frac{\partial B(t,x)}{\partial t} \tag{2.3}$$

We obtain as follows by using a discrete external force.

$$\frac{\partial \bar{C}(t,x)}{\partial t} = A_x \bar{C}(t,x) + \sum_j \varphi(x,x^j)f_j(t) + d_n\frac{\partial B(t,x)}{\partial t} \tag{2.4}$$

The stochastic unique solution for Equations (2.3) and (2.4) can be proved by using the above-mentioned distribution disturbance[7]. Next, The observation system is presented as follows:

$$y(t) = \boldsymbol{C}(x,x^j)\{C(t,x^j)\} \equiv \boldsymbol{C}(x,x^j)\{C_j(t)\} \tag{2.5}$$

Assumption 2.1 *Uniform input space in Equation (2.3) for simplicity*

According to Assumption 2.1, we obtain as follows:

$$\frac{\partial \bar{C}(t,x)}{\partial t} = A_x C(t,x) + b(x)f(t) + \varepsilon(x,x^{'})\xi_x(t) \tag{2.6}$$

where $\varepsilon(x,x^{'})$ is a Generalized random variable.

Then, we derive as follows to represent the meaning of the Generalized random variable.

$$\frac{\partial \bar{C}(t,x)}{\partial t} = A_x C(t,x) + b(x)f(t) + \xi^{\varepsilon}(t) \tag{2.7}$$

where $\xi^{\varepsilon}(t) = \xi(t,x)(\varphi(x,x^{'})\xi_x(t))$ as $\varepsilon \to 0$.

3 Optimal servo system for the linear distributed parameter systems

3.1 Equivalent distribution disturbance

We present the system model for linear distributed parameter systems again.

$$\begin{aligned} \frac{\partial C(t,x)}{\partial t} &= A_x C(t,x) + b_x f(t) + b_{dx}\xi(t) \\ y &= \boldsymbol{C}\{C\}(t,x), \quad x \in \Omega \end{aligned} \tag{3.1}$$

where The observation system is as follows.

$$y(t,x) = \int_{\Omega^{'}} \boldsymbol{C}(t,x)\{C(t,x)\}d\Omega^{'}, \quad \Omega^{'} \in \Omega \tag{3.2}$$

Definition 3.1 *Equivalent distribution disturbance*

$$\bar{b}_x\bar{\xi}(t) = \int_{\Omega^{''}} G(x)b_{dx}\xi(t)d\Omega^{''}, \quad \Omega^{''} \in \Omega \tag{3.3}$$

where summing that $\bar{b}_x \simeq b_x$.

Then,

$$\begin{aligned} \frac{\partial \bar{C}(t,x)}{\partial t} &= A_x \bar{C}(t,x) + b_x f(t) + b_x\bar{\xi}(t) \\ \bar{y} &= \boldsymbol{C}\{\bar{C}\}(t,x) \end{aligned} \tag{3.4}$$

where $\bar{y}(t,x)$ is derived as follows:

$$\bar{y}(t,x) = \int_{\Omega^{'}} \boldsymbol{C}(t,x)\{\bar{C}(t,x)\}d\Omega^{'}, \quad \Omega^{'} \in \Omega \tag{3.5}$$

With respect to the equivalent distribution disturbance $\bar{\xi}$, $\bar{\xi}$ is called equivalent distribution disturbance when the following conditions are satisfied.

1. Assumption of equivalence between control input distribution and disturbance distribution

$$b_x \equiv \bar{b}_x \tag{3.6}$$

2. The following equation is satisfied under Equation (3.6).

$$y(t,x) = \bar{y}(t,x), \quad \bar{b}_x = \bar{\xi}(t) \tag{3.7}$$

Then, assuming that $G(x)$ exists.

$$\bar{b}_x\bar{\xi}(t) = \int_{x\in D} G(x) b_{dx}\xi(t)dx \tag{3.8}$$

According to Figure 3, we obtain the mathematical model of $\hat{C}(t,x)$ as follows:

$$\frac{\partial \hat{C}(t,x)}{\partial t} = A_x\hat{C}(t,x) + b_x f(t) + \boldsymbol{LC}[C-\hat{C}] \tag{3.9}$$

We obtain the mathematical model of the estimation variable $\tilde{C}$ for the true value of the equivalent distribution disturbance $\bar{\xi}(t,x)$ as follows:

$$\frac{\partial \tilde{C}(t,x)}{\partial t} = A_x\tilde{C}(t,x) + b_x[f(t)+\bar{\xi}], \quad \tilde{C} = \hat{C} - C_e \tag{3.10}$$

Further,

$$\frac{\partial \tilde{C}(t,x)}{\partial t} = A_x\tilde{C}(t,x) + b_x f_u + b_x\left[\frac{\delta C_e}{\delta t} - A_x C_e\right] + b_x\bar{xi} \tag{3.11}$$

where C_e satisfy as follows:

$$\frac{\delta C_e}{\delta t} = A_x C_e + \delta\hat{\xi}, \quad \hat{\xi} = \bar{\xi} + \delta\hat{\xi} \tag{3.12}$$

We obtain by using Equations (3.11) and (3.12) as follows:

$$\frac{\partial \hat{C}(t,x)}{\partial t} = A_x\hat{C}(t,x) + b_x[f_u + \hat{\xi}] \tag{3.13}$$

Moreover, from Equation (3.9), we obtain as follows:

$$\frac{\partial \hat{C}(t,x)}{\partial t} = A_x\hat{C}(t,x) + b_x f_u^* + \boldsymbol{LC}[C-\hat{C}] \tag{3.14}$$

3.2 Optimal input for the for linear distributed parameter systems

We obtain as follows by using Equations (3.13) and (3.14).

$$A_x\hat{C}(t,x) + b_x f_u^*(t) + \boldsymbol{LC}[C-\hat{C}] = A_x\hat{C}(t,x) + b_x(f_u+\hat{\xi}), \quad f_u^* = -K_p\hat{C} \tag{3.15}$$

We could obtain the optimal input $f_u^* = -K_p\hat{C}$.

Next, we discuss the distribution filter. We obtain as follows by modifying Equation (3.15).

$$\hat{\xi} = (f_u^* - f_u) + \frac{1}{b_x}\boldsymbol{LC}(C-\hat{C}) \tag{3.16}$$

It assumes the existence of operator in the relationship of $\hat{\xi}$ and $\tilde{\xi}$.

Assumption 3.1 *Existence of operator* $g_{t,x}(\bullet)$

From Assumption 3.1, we obtain as follows:

$$\tilde{\xi} = g_{t,x}[\hat{\xi}] \tag{3.17}$$

where $g_{t,x}(\bullet)$ is called as a linear distribution filter[13]. For example, $g_{t,x}(\bullet)$ is considered as follows:

$$g_{t,x}(\bullet) \equiv \left(\frac{\partial}{\partial t} + K\frac{\partial}{\partial x}\right)^n(\bullet), \quad K>1 \quad and \quad n>1 \tag{3.18}$$

Here, $f = f_u^* - \tilde{\xi}$. We replace $g_{t,x}(\bullet)$ as follows:

$$\tilde{\xi} = F_\xi[\hat{\xi}], \quad F_\xi(\bullet) \equiv g_{t,x}(\bullet) \tag{3.19}$$

We assume the following equation.

Assumption 3.2

$$||\hat{\xi} - \tilde{\xi}|| < ||\hat{\xi}|| \tag{3.20}$$

In order to break down equation (3.20) further, we describe the equation as follows:

$$\begin{aligned}\frac{\partial \hat{C}(t,x)}{\partial t} &= A_x\hat{C}(t,x) + b_x f(t) + \boldsymbol{LC}[C - \hat{C}] \\ f &= f_u^* - \tilde{\xi} \\ \delta C &= \hat{C} - C\end{aligned} \tag{3.21}$$

In case of $r = 0$ and $\xi = 0$, we obtain as follows:

$$\frac{\partial \hat{C}(t,x)}{\partial t} = A_x C(t,x) + b_x f(t) \tag{3.22}$$

We analyze the control input stability conditions. From Equation (3.9), we obtain as follows:

$$\begin{aligned}\frac{\delta \hat{C}(t,x)}{\partial t} &= [A_x - \boldsymbol{LC}]\delta + b_x\tilde{\xi} \\ &= [A_x(\delta C) - \int_{x\in\Omega} L(\Omega,t)\boldsymbol{C}(\Omega)\delta C(\Omega,t)d\Omega]\end{aligned} \tag{3.23}$$

At that time, the following equation is satisfied.

$$||\hat{\xi} - \tilde{\xi}|| < ||\hat{\xi}|| \tag{3.24}$$

Here, $\hat{\xi}$ is as follows:

$$\begin{aligned}\hat{\xi} &= -\frac{1}{b_x}\boldsymbol{LC}[C - \hat{C}] + f_u^* - f = -\frac{1}{b_x}\boldsymbol{LC}[\delta C] + f_u^* - f \\ &= -\frac{1}{b_x}\boldsymbol{LC}[\delta C] + \hat{\xi} \\ &= -\frac{1}{b_x}\int_{x\in\Omega} L(\Omega,t)\boldsymbol{C}(\Omega)[\delta C(\Omega,t)]d\Omega + \tilde{\xi}(\Omega,t)\end{aligned} \tag{3.25}$$

From Figure 4, $\Omega(\bullet$ is the general operator from $\hat{\xi}$ to $\tilde{\xi}$ According to the Small Gain Theory[14], we obtain as follows:

$$\begin{aligned}||\hat{\xi} - \tilde{\xi}|| &= ||\Omega(\tilde{\xi}) - \Omega(\hat{\xi})|| \\ &= ||\Omega(\tilde{\xi}) - F_\xi \cdot \Omega(\tilde{\xi})||\end{aligned} \tag{3.26}$$

From Equation (3.24), we obtain as follows:

$$||[\Omega - F_\xi \cdot \Omega]|| < ||\Omega|| \tag{3.27}$$

From Equation (3.27), we obtain as follows:

$$||F_\xi \cdot \Omega|| < \boldsymbol{I} \tag{3.28}$$

Therefore, the system is stable if Equation (3.28) is satisfied for the optimal gain at the optimal input f_u^*. However, f_u satisfies the following equation.

$$f_u(t,x) = f_u^*(t,x) - \hat{\xi}(t,x) \tag{3.29}$$

With respect to the description of the partial differential equation for Equations (3.9), (3.10), (3.13), (3.14), (3.15) and (3.16), please refer to the Appendix B.

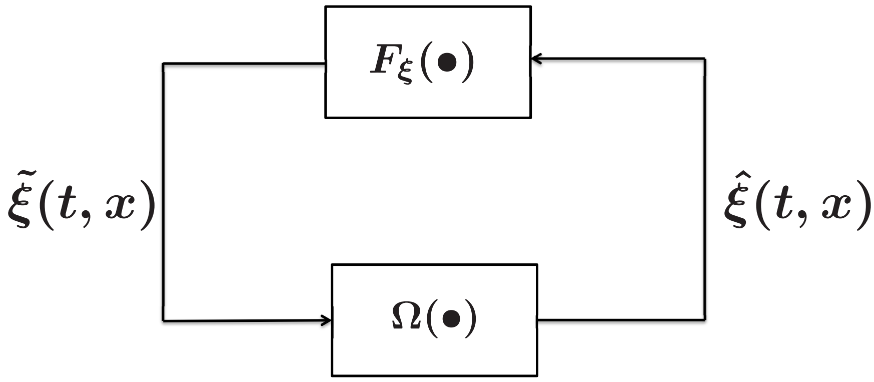

Fig. 4: Relationship between $\hat{\xi}$ and $\tilde{\xi}$

4 Results

We have reported the configuration of a zero-balance regulator system or a servo system with a disturbance against a linear distributed parameter system. We introduced a generalized random variable theory to handle the independent Wiener process for each point in space. The spatial distribution white noise could be approximated by a sufficiently smooth function from the regularization concept used in a generalized random variable theory. Next time, we will report on the observer of a distributed parameter system with distribution noise.

References

[1] Akira Kaneko: Introduction of Partial differential equations; University of Tokyo Press, 1998

[2] K.Shirai and Y.Amano: Production density diffusion equation propagation and production; IEEJ Transactions on Electronics, Information and Systems, 2012, vol.132-C, no.6, p.983-990

[3] Shima, M, Ikeda, S. et al.: Optimal Control of a Bilinear Distributed Parameter System -On the Pattern of the Optimal Flow-Rate Control-; Journal of ISCIE, Vol. 17, No- 11, pp.689–697

[4] H.Tasaki: Thermodynamics; Maruzen co., LTD, 1998

[5] Kenji Shirai and Yoshinori Amano, etc: Mathematical Model of Thermal Reaction Process for External Heating Equipment in the Manufacture of Semiconductors(Part1); International Journal of Innovative Computing, Information and Control, 2013, vol.9, no.4, p.1557-1571

[6] Kenji Shirai and Yoshinori Amano, etc: Mathematical Model of Thermal Reaction Process for External Heating Equipment in the Manufacture of Semiconductors(Part2); Internatonal Journal of Innovative Computing, Information and Control, 2013, vol.9, no.5, p.1889-1898

[7] Kenji Shirai and Yoshinori Amano: Model of Production System with Time Delay using Stochastic Bilinear Equation; Asian Journal of Management Science and Applications , Vol. 12, No. 1, pp.83-103, 2015

[8] M. Corless and J. Tu: State and input estimation for a class of uncertain systems; Automatica, Vol.34, Issue.6, p.757-764, 1998

[9] J.She and Y.Ohyama, etc: Improvement of disturbance rejection performance by equivalent input disturbance estimation; Journal of SICE, Vol.41, No.10, pp.797-802, 2005

[10] H.J. Kushuner: On the optimal control of a system governed by a linear parabolic equation with white noise input; SIAM J. Contrl, Vol.6, Issue 4, pp.346-359, 1968

[11] S.G. Tzafestas and J.M. Nightingale: Optimal control of a class of linear stochastic distributed parameter systems; Proc IEE, Vol.115, pp.1213-1220, 1968

[12] Y.Sunahara and K.Kamejima: Mathematical model of spatially distributed white gaussian process; The 14th Symposium on Statistical Control Theory, pp.47-52, 1973

[13] S.Hata, H.Shibata and S.Ohmatu: The Stochastic differential equation in the infinite dimensional space and Its application to the filtering problems; The 14th Symposium on Statistical Control Theory, pp.47-52, 1973

[14] M.Sanpei: Exact linearization and its application to the trajectory control of the towing vehicle; The society of Instrument and control engineer, vol.31, pp.851-858, 1992

[15] J.G.Truxal: Automatic feedback control system synthesis; McGraw-Hill Book Company, NY, 1955

A Example of Friedrichs's mollifier operator

Friedrichs's mollifier operator satisfies the following items.

- A bounded area D
- The following equation should be satisfied.

$$\int_{R^n} \rho(x)dx = 1$$

- The following equation should be satisfied.

$$\lim_{\varepsilon\to 0} \rho^{\varepsilon}(x) = \lim_{\varepsilon\to 0} e^{-n}\rho(\frac{x}{\varepsilon}) = \delta(x)$$

The example of Friedrichs's mollifier operator ρ is as follows:

$$\rho(x) = \begin{cases} \exp(-\frac{1}{1-|x|^2)} & |x| < 1 \\ 0 & |x| \leq 1 \end{cases}$$

The correlation functional is derived as follows:

$$\Phi^{\varepsilon}(f)(x) = \int_{R^n} \rho^{\varepsilon}(x-y)f(y)dy, \quad \rho^{\varepsilon}(x) = e^{-n}\rho\left(\frac{x}{\varepsilon}\right)$$

B Description of the partial differential equation for Equations (3.9), (3.10), (3.13), (3.14), (3.15) and (3.16)

Equation (3.9) is derived by the partial differential equation as follows:

$$\frac{\partial \hat{C}}{\partial t} = A_x \hat{C} + b_x(\Omega) f(t) + \int_{x\in\Omega} L(t,\Omega)\boldsymbol{C}(\Omega)C(t,\Omega)d\Omega - \int_{x\in\Omega} L(t,\Omega)\boldsymbol{C}(\Omega)\hat{C}(t,\Omega)d\Omega \tag{B.1}$$

Equation (3.10) is derived by the partial differential equation as follows:

$$\frac{\partial \tilde{C}}{\partial t} = A_x \tilde{C} - \int_{x\in\Omega} b_x(\Omega) K(t,\Omega^{'})[\hat{C}(t,\Omega) + C(t,\Omega)]d\Omega \tag{B.2}$$

Equation (3.13) is derived by the partial differential equation as follows:

$$\frac{\partial \hat{C}}{\partial t} = A_x \hat{C} + \int_{x\in\Omega} b_x(\Omega)\left[-\int_{x\in\Omega} K(t,\Omega)\{\hat{C}(t,\Omega) + C(t,\Omega)\}d\Omega\right] + \int_{x\in\Omega} b_x(\Omega)\hat{\xi}(t,\Omega)d\Omega \tag{B.3}$$

Equation (3.14) is derived by the partial differential equation as follows:

$$\begin{aligned}\frac{\partial \hat{C}}{\partial t} &= A_x \hat{C} + \int_{x\in\Omega} b_x(\Omega) f_u^*(t,\Omega)d\Omega + \int_{x\in\Omega} L(t,\Omega)\boldsymbol{C}(\Omega)C(t,\Omega)d\Omega \\ &\quad - \int_{x\in\Omega} L(t,\Omega)\boldsymbol{C}(\Omega)\hat{C}(t,\Omega)d\Omega\end{aligned} \tag{B.4}$$

Equation (3.15) is derived by the partial differential equation as follows:

$$\begin{aligned}&A_x \hat{C} + \int_{x\in\Omega} b_x(\Omega) f_u^*(t,\Omega)d\Omega + \int_{x\in\Omega} L(t,\Omega)\boldsymbol{C}(\Omega)C(t,\Omega)d\Omega - \int_{x\in\Omega} L(t,\Omega)\boldsymbol{C}(\Omega)\hat{C}(t,\Omega)d\Omega \\ &= A_x \hat{C} + \int_{x\in\Omega} b_x(\Omega) f_u(t,\Omega)d\Omega + \int_{x\in\Omega} b_x(\Omega)\hat{\xi}(t,\Omega)d\Omega\end{aligned} \tag{B.5}$$

Equation (3.16) is derived by the partial differential equation as follows:

$$\begin{aligned}\hat{\xi}(t,\Omega) &= \int_{x\in\Omega}\left[f_u^*(t,\Omega) - f_u(t,\Omega)\right]d\Omega \\ &\quad + \int_{x\in\Omega}\frac{1}{b_x(\Omega)}\left[L(t,\Omega)\boldsymbol{C}(\Omega)C(t,\Omega)\right]d\Omega - \int_{x\in\Omega}\frac{1}{b_x(\Omega)}\left[L(t,\Omega)\boldsymbol{C}(\Omega)\hat{C}(t,\Omega)\right]d\Omega\end{aligned} \tag{B.6}$$