A formal theory of the calculus of indication

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Abstract

This paper deals with a term reduction representation of the calculus of indication proposed by G. Spencer-Brown's Laws of Form, which has a formalism of great simplicity for the act of distinguishing and its basic laws. I will give an equational theory based on the term reduction of indication in order to make an interpretation of this calculus more explicit way.

1 Introduction

In [8], G.Spencer-Brown proposed the calculus of indication which was firstly intended to give mathematical basics for Boolean algebra of logic. In fact, since Frege and Russell's Principia Mathematica, it has been taken by right that one could not find more simple basics for logic than the notion of true and false as valued of simple statements. On the ground that Boolean algebra designed to fit logic with the above basics has not any mathematical interest about their arithemetics, alternatively G.Spencer-Brown employed a geometrical formal system based on a primitive act (rather than a logical value) of distinguishing a space to duality (outside and inside) like the skin of a living organism cuts off in the same way. What is the difference of two basics, is that for example give a statement to be analyzed, the analysis need not stop at the point where its truth value is assigned, but the statement has more deeper content on the act of distinction and its indication. Namely, every logically equivalent statements are not necessarily identical in the situations arised by the act of distinction. Here is a brief description of his calculus (see [9] and [12] for details).

G. Spencer-Brown explored the indications arising from the act of distinguishing, that is simply identified with the *name* of the content of the distinction. In his representation of indication, it assumed that all distictions and all its domains, i.e., all spaces are alike, respectively. So by erasing every qualitative difference of the distinctions, we can reduce them to their basic quality of generating a boundary in whatever domain. This gives rise to the notion of *primary* distinction and *indicational space,* and to consider calculations among them. The exploration was inspired by establishing the following:

Definition 1.1 *Distinction is perfect continence.*

Axiom 1.2

($A1$) The law of calling

The value of a call made again is the value of the call.

($A2$) The law of crossing

The value of a crossing made again is not the value of the crossing.

The above definition means that the act of distinction can be done by arranging a boundary with separate sides. For example, drawing a circle in a plane is a distinction. The first axiom says that to refer (or call) a situation of distinction repeatedly is the same virtue as a single reference (or calling). Also, the second axiom says that the act of distinction in twice is a void. So there are two kind of operations in the reference of situations of distinction, that is, juxtaposition and a king of exponentiation. Now we employ Ø and $\boldsymbol{D}()$ as the sign of a void

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space and an act of distinction (or its indication), respectively. Then the sign $\boldsymbol{D}()$ has the operator-operand polarity, that is, the distinction $\boldsymbol{D}()$ represents not only an act of distinction (as an operator) but also a situation of distinction (as an operand). In order to account for the implicit operations of distinction and the polarity we clarify the reduction process for the calculus of indication by introducing a formal theory of indicational equality.

In section 2, we recall the primary arithmetic in G. Spencer-Brown's book [9]. Then we define ***CI***-terms on the indicational language $\boldsymbol{L}_{\mathrm{CI}}$ consisting of a set of variables, a constant $\emptyset$ and unary function $\boldsymbol{D}$, and also a formal theory ***CI*** of I-equality that is corresponding to the primary arithmetic. Futhermore, each theorem involved with the primary arithmetic is revised on ***CI***. In section 3, we introduce an algebraic theory ***PA*** of I-equality corresponding to the primary algebra, and revise their results. In section 4, we demonstrate an interpretation of ***PA*** within the classical propositional logic by guiding principle of appendix 2 in [9]. It is important to note that the calculus of indication has many possible interpretations beyond the classical propositional logic. In final section, we summarize several isomorphisms between ***PA*** and Boolean algebra with some distinct primitives based on results proposed before now, and discuss also some remaining problems and further subjects.

2 The primary arithmetic

2.1 Recalling the primary arithmetic

The following definitions are the short summary of G. Spencer-Brown's, *Laws of Form* (see [9]).

Definition 2.1 *The form is generated by drawing a* distinction. *Call it* the first distinction. *Call the space in which it is drawn* the space severed by the distinction. *Call the parts of the space shaped by the severance or, alternatively,* the spaces, states, or contents distinguished by the distinction.

Definition 2.2 *Let any* mark, token, or sign *be taken with regard to the distinction as a signal. Call the use of any signal* its intent. *Let a state distinguished by the distinction be* marked with a mark $\boldsymbol{D}()$ *of distinction (we employ* $\boldsymbol{D}()$ *instead of the original cross). Call the state* the marked state. *Call the state not marked with the mark* the unmarked state *(we specify this by* $\emptyset$*).*

Definition 2.3 *Call the space severed by any distinction, together with the entire content of the space,* the form of the distinction. *Call the form of the first distinction* the form.

Definition 2.4 *Call any copy of the mark a* token *of the mark. Let any token of the mark be called as a* name *of the marked state. Let the name indicate the state.*

Definition 2.5 *Call the form of a number of tokens considered with regard to one another an* arrangement. *Call any arrangement intended as an indicator an* expression. *Call a state indicated by an expression the* value *of the expression. Call expression of the same value* equivalent. *(Let a sign* = *of equivalence be written between equivalent expressions.)*

Definition 2.6 *Let any token be taken for intention. Let any token be given a name* cross *to indicate what the intention is. Let each token of the mark be seen to cleave the space into which it is copied. That is to say, let each token be a distinction in its own form. Call the concave side of a token its* inside. *Let any token be intended as an instruction to cross the boundary of the first distinction. Let the crossing be from the state indicated on the inside of the token to the state indicated by the token. Let a space with no token indicate the unmarked state.*

Definition 2.7 *The form of every token called cross is to be perfectly continent. We have allowed only one kind of relation between crosses. Let the intent of this relation be restricted so that a cross is said to contain what is on its inside and not to contain what is not on its inside.*

Definition 2.8 *Call an indication of equivalence expressions an* equation. *Call the following two equations* primitive.

(*P*1) $\boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset) = \boldsymbol{D}(\emptyset)$ *(by A1)*

(*P*2) $\boldsymbol{D}(\boldsymbol{D}(\emptyset)) = \emptyset$ *(by A2)*

Definition 2.9 *Call any expression consisting of an empty token* simple *(i.e,* $\boldsymbol{D}(\emptyset)$*). Call any expression consisting of an empty space* simple *(i.e,* $\emptyset$*). Let there be no other simple expression.*

2.2 CI-term and its substitution

I will introduce a term reduction representation of the calculus of indication in the same manner of λ-calculus ([2],[4]). Let $\boldsymbol{L}_{\mathrm{CI}} = \langle \mathrm{L}_{\mathrm{CI}}, \boldsymbol{D}, \emptyset \rangle$ be the indicational language consisting of infinite denumerable set of variables e_1, e_2, e_3,..., a constant $\emptyset$ (unmarked state), unary function $\boldsymbol{D}$ (indicator or marked state). Then we have the following.

Definition 2.10 (*i*) *The set of* ***CI***-terms *I is defined inductively as follows:*

(1) *All variables and constant* $\emptyset$ *are* ***CI****-terms (called* atoms*).*

(2) *If M and N are any* ***CI****-terms, then (MN) is a* ***CI****-term (called a* calling*).*

(3) *If M is any* ***CI****-term, then* ***D****(M) is a* ***CI****-term (called a* distinction*).*

(4) *Nothing is a* ***CI****-term except as required by (1),(2) and (3).*

(*ii*) *Call any* ***CI****-terms with no variables* the closed ***CI***-terms.

Let *M, N, L,...* denote arbitary ***CI***-terms. $M_1M_2M_3\cdots M_n$ is an abbreviation of $((\cdots((M_1M_2)M_3)\cdots)M_n)$, where it assumes that associative and commutative laws for parentheses hold. Also $\boldsymbol{D}(M)N$ is an abbreviation of $(\boldsymbol{D}(M))N$. We employ the symbol $\equiv$ to denote syntactic equality. For example, $\emptyset$, $\boldsymbol{D}(\emptyset)$ and $MN\boldsymbol{D}(\boldsymbol{D}(L)\boldsymbol{D}(\boldsymbol{D}(\emptyset)P)R)$ are all ***CI***-terms.

Definition 2.11 *Let M be any* ***CI****-term. Then the* depth *of a term M (notation* dph(*M*)*) is the total number of nesting occurrences of* ***D*** *in M. More precisely, can be defined inductively as follows:*

(1) *dph(a)=0 for any atomic term a.*

(2) *dph(MN)=max{dph(M),dph(N)}*

(3) *dph(****D****(M))=1+dph(M)*

For example, we have dph($\emptyset$)=0, dph($\boldsymbol{D}(a)b$)=1 and dph($\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(a)b))$)=3.

Definition 2.12 *M is a* subterm *of N(notation* $M \subset N$*) if* $M \in Sub(N)$*, where Sub(N), the collection of subterm of N, is defined inductively as follows:*

(*i*) $Sub(a)=\{a\}$ *for any atomic term a.*

(*ii*) $Sub(MN)=Sub(M)\cup Sub(N)\cup \{MN\}$

(*iii*) $Sub(\boldsymbol{D}(M))=Sub(M)\cup \{\boldsymbol{D}(M)\}$

A subterm may occur several times; $M \equiv \boldsymbol{D}(\boldsymbol{D}(N))\boldsymbol{D}(N)$ has two occurrences of the subterm $\boldsymbol{D}(N)$. Let N_1, N_2 be subterm occurrences of *M*. Then N_1, N_2 are *disjoint* if N_1 and N_2 have no common symbol occurrences.

Definition 2.13 *Let M be any **CI**-term. Then the iteration of* distinction *is defined inductively as follows:*

(1) $\boldsymbol{D}^0(M) \equiv M$

(2) $\boldsymbol{D}^{n+1}(M) \equiv \boldsymbol{D}(\boldsymbol{D}^n(M))$

Definition 2.14 *Let M be any **CI**-term. Then the iteration of* calling *is defined inductively as follows:*

(1) $0M \equiv \emptyset$

(2) $(n+1)M \equiv nMM$

For example, we have the abbreviations: $\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\emptyset))) \equiv \boldsymbol{D}^3(\emptyset)$, $aaabb \equiv 3a2b$ or $a2(ab)$ for any atoms a, b and $a\boldsymbol{D}(pq)\boldsymbol{D}(pq) \equiv a2\boldsymbol{D}(pq)$.

Definition 2.15 *Let any **CI**-term M has the* space pervading it. *Let the space of a term N is* one more deeper *than the space of a term M if* $M \equiv \boldsymbol{D}(N)$. *Let M be any **CI**-term. Then call the space of term M* the shallowest space *with regard to term M. Let M be any **CI**-term. Then call the space of depth* dph(M) the deepest space *with regard to term M. Let any indicator* $\boldsymbol{D}_m$ *standing in any space in a indicator* $\boldsymbol{D}_n$ $(n < m)$ *be said to* be contained *in* $\boldsymbol{D}_n$. *Let any indicator* $\boldsymbol{D}_{n+1}$ *standing in any space in a indicator* $\boldsymbol{D}_n$ *be said to stand under the indicator* $\boldsymbol{D}_n$. *Let M be any **CI**-term. Then each subterm of M is pervaded by any space under the depth* dph(M).

Definition 2.16 *For any **CI**-term M,N, define* $[N/X]M$ *to be the result of* substituting *N for any subterm X in M inductively as follows:*

(1) $[N/X]X \equiv N$

(2) $[N/X]P \equiv P$ *if* $X \not\equiv P$

(3) $[N/X](PQ) \equiv ([N/X]P[N/X]Q)$

(4) $[N/X]\boldsymbol{D}(P) \equiv \boldsymbol{D}([N/X]P)$ *if* $X \not\equiv \boldsymbol{D}(P)$

2.3 I-reduction

Definition 2.17 *For the set of **CI**-terms I, let* I^- *be a notion of* simplification *on I.*

(*i*) *The* simplification I^- *has the following three binary relations.*

(1) $\rightharpoonup$ *(or* $\rightarrow_{I^-}$*) : one step* I^-*-reduction*

(2) $\rightharpoonup\!\!\!\rightharpoonup$ *(or* $\twoheadrightarrow_{I^-}$*) :* I^-*-reduction*

(3) $=_{I^-}$: I^-*-equality*

(*ii*) *One step* I^--reduction *is defined inductively as follows:*

(1) $\boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset) \rightharpoonup \boldsymbol{D}(\emptyset)$ *(Condensation)*

(2) $\boldsymbol{D}(\boldsymbol{D}(\emptyset)) \rightharpoonup \emptyset$ *(Cancellation)*

(3) $M\emptyset \rightharpoonup M$

(4) $MN \rightharpoonup NM$

(5) $M(NL) \rightharpoonup (MN)L$

(6) $LM \rightharpoonup LN \Rightarrow M \rightharpoonup N$

(7) $\boldsymbol{D}(M) \rightharpoonup \boldsymbol{D}(N) \Rightarrow M \rightharpoonup N$

(*iii*) $\rightharpoonup\!\!\!\rightharpoonup$ *is the reflexive, transitive closure of* $\rightharpoonup$ *:*

(*iv*) $=_{I^-}$ *is the equivalence relation generated by* $\rightharpoonup\!\!\!\rightharpoonup$ *:*

Definition 2.18 *For the set of **CI**-terms I, let* I^+ *be a notion of* complication *on I.*

(*i*) *The* complication I^+ *has the following three binary relations.*

(1) $\rightharpoondown$ *(or* $\rightarrow_{I^+}$ *) : one step* I^+*-reduction*

(2) $\rightharpoondown\!\!\!\rightharpoondown$ *(or* $\twoheadrightarrow_{I^+}$ *) :* I^+*-reduction*

(3) $=_{I^+}$: I^+*-equality*

(*ii*) *One step* I^+-reduction *is defined inductively as follows:*

(1) $\boldsymbol{D}(\emptyset) \rightharpoondown \boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset)$ *(Confirmation)*

(2) $\emptyset \rightharpoondown \boldsymbol{D}(\boldsymbol{D}(\emptyset))$ *(Compensation)*

(3) $M \rightharpoondown M\emptyset$

(4) $MN \rightharpoondown NM$

(5) $M(NL) \rightharpoondown (MN)L$

(6) $M \rightharpoondown N \Rightarrow LM \rightharpoondown LN$

(7) $M \rightharpoondown N \Rightarrow \boldsymbol{D}(M) \rightharpoondown \boldsymbol{D}(N)$

(*iii*) $\twoheadrightarrow$ *is the reflexive, transitive closure of* $\rightharpoondown$

(*iv*) $=_{I^+}$ *is the equivalence relation generated by* $\twoheadrightarrow$

Definition 2.19 *For the set of* ***CI****-terms I, let* I^* *be a notion of* calculation *on I.*

(*i*) *The* calculation I^* *has the following three binary relations.*

(1) $\rightarrow$ *(or* $\rightarrow_{I^*}$*) : one step* I^**-reduction*

(2) $\twoheadrightarrow$ *(or* $\twoheadrightarrow_{I^*}$*) :* I^**-reduction*

(3) $=_{I^*}$: I^**-equality (or we also simply employ = : equality)*

(*ii*) *One step* I^*-reduction *is defined as follows:*

$$M \rightarrow N \overset{\triangle}{\Longleftrightarrow} M \rightharpoonup N \text{ or } M \rightharpoondown N$$

(*iii*) $\twoheadrightarrow$ *is the reflexive, transitive closure of* $\rightarrow$

(*iv*) $=_{I^*}$ *is the equivalence relation generated by* $\twoheadrightarrow$

Definition 2.20 (*i*) $\boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset)$ and $\boldsymbol{D}(\boldsymbol{D}(\emptyset))$ *are called a* ***CI***-redex, *and the corresponding terms* $\boldsymbol{D}(\emptyset)$, $\emptyset$ *are called its* ***CI***-contractum.

(*ii*) *A term M which contains no* ***CI****-redex is called a* ***CI***-normal form *(or* simple term*).*

(*iii*) *The class of all* ***CI****-normal form is called* ***CI****-nf. If a term M* I^**-reduces to a N in* ***CI****-nf, then N is called a* ***CI****-normal form of M.*

(*iv*) *If there is a* I^**-reduction from a term M to its* ***CI****-nf N, then call this* I^**-reduction the* calculation *of M.*

Definition 2.21 (CI, the formal theory of I^*-equality)

(*i*) *The formulas of* ***CI*** *are just M = N, for all* ***CI****-terms M, N. This theory is axiomatized by the following axioms and rules:*

(A1) $2\boldsymbol{D}(\emptyset) = \boldsymbol{D}(\emptyset)$ *(Number)*

(A2) $\boldsymbol{D}^2(\emptyset) = \emptyset$ *(Order)*

(A3) $M = M$ (*)†

(A4) $MN = NM$

(A5) $M(NL) = (MN)L$

(A6) $M = M\emptyset$

(R1) $\dfrac{M = N}{ML = NL}$

(R2) $\dfrac{M = N}{\boldsymbol{D}(M) = \boldsymbol{D}(N)}$

(R3) $\dfrac{M = N}{N = M}$ (*)

† In precisely speaking, these axiom and rules are not necessary because they are provable in *CI*, see theorem 2.29-31.

$$(R4)\ \frac{M = N \quad N = L}{M = L} \qquad (*)$$

(*ii*) *Provability in* ***CI*** *of an equation is denoted by* ***CI*** $\vdash M = N$ *or often just by* $M = N$. *If* ***CI*** $\vdash M = N$, *then* M *and* N *are called* I*-convertible.

(*iii*) *Call also this theory* the primary arithmetic.

2.4 Primary arithmetic

Theorem 2.22 *A* ***CI****-term consisting of a finite number of indicators can be simplified to a simple term.*

Proof We show this by induction for the number of nesting indicators of a term. Consider any *CI*-term M with a finite number of $\boldsymbol{D}()$ in space s. Then there exits a natural number n such that $n = dph(M)$. By the definition, the shallowest space of M is $s = s_0$ and the deepest space of M is s_n.

(1) $n = 0$

By $s_n = s_0 = s$, we get a simple term $M \equiv \emptyset$.

(2) $n = 1$

(*i*) $M \equiv \boldsymbol{D}(\emptyset)$

This case is already a simple term.

(*ii*) $M \equiv \boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset)\cdots\boldsymbol{D}(\emptyset) \equiv m\boldsymbol{D}(\emptyset)\ (m \in N)$

$$\begin{aligned} m\boldsymbol{D}(\emptyset) &\rightharpoonup (m\text{-}1)\boldsymbol{D}(\emptyset) \\ &\rightharpoonup (m\text{-}2)\boldsymbol{D}(\emptyset) \\ \cdots &\rightharpoonup \boldsymbol{D}(\emptyset) && (A1) \end{aligned}$$

Hence, this case is also a simple term.

(3) For $n \le k$, assume that the *CI*-term M can be simplified to a simple term. Then consider a *CI*-term M with depth $n = k{+}1$.

(*i*) $M \equiv \boldsymbol{P}\boldsymbol{D}_k(\boldsymbol{D}_{k+1}(\emptyset))$ (where $dph(P) < k$)

$$\begin{aligned} \boldsymbol{P}\boldsymbol{D}_k(\boldsymbol{D}_{k+1}(\emptyset)) &\rightharpoonup \boldsymbol{P}\emptyset && (A2) \\ &\rightharpoonup P && (A6) \\ &\rightharpoonup \boldsymbol{D}(\emptyset) \text{ or } \emptyset && \text{(I.H.)} \end{aligned}$$

(*ii*) $M \equiv \boldsymbol{P}\boldsymbol{D}_k(m\boldsymbol{D}_{k+1}(\emptyset))$ (where $m \in N$, $dph(P) < k$)

$$\begin{aligned} \boldsymbol{P}\boldsymbol{D}_k(m\boldsymbol{D}_{k+1}(\emptyset)) &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_k((m\text{-}1)\boldsymbol{D}_{k+1}(\emptyset)) \\ &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_k((m\text{-}2)\boldsymbol{D}_{k+1}(\emptyset)) \\ &\ \vdots \\ &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_k(\boldsymbol{D}_{k+1}(\emptyset)) && (A1) \\ &\rightharpoonup \boldsymbol{P}\emptyset && (A2) \\ &\rightharpoonup \boldsymbol{D}(\emptyset) \text{ or } \emptyset && \text{(I.H.)} \end{aligned}$$

(*iii*) $M \equiv \boldsymbol{P}\boldsymbol{D}_{k\text{-}1}(n\boldsymbol{D}_k(m\boldsymbol{D}_{k+1}(\emptyset)))$ (where m, $n \in N$, $dph(P) < k$)

$$\begin{aligned} \boldsymbol{P}\boldsymbol{D}_{k\text{-}1}(n\boldsymbol{D}_k(m\boldsymbol{D}_{k+1}(\emptyset))) &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_{k\text{-}1}(\boldsymbol{D}_k((m\text{-}1)\boldsymbol{D}_{k+1}(\emptyset))) \\ &\ \vdots \\ &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_{k\text{-}1}(n\boldsymbol{D}_k(\boldsymbol{D}_{k+1}(\emptyset))) && (A1) \\ &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_{k\text{-}1}(n\emptyset) && (A2) \\ &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_{k\text{-}1}((n\text{-}1)\,\emptyset) && (A6) \\ &\ \vdots \\ &\rightharpoonup \boldsymbol{P}\boldsymbol{D}_{k\text{-}1}(\emptyset) && (A6) \\ &\rightharpoonup \boldsymbol{D}(\emptyset) \text{ or } \emptyset && \text{(I.H.)} \end{aligned}$$

Thereby, every $\boldsymbol{CI}$-term M with a finite number of indicators can be simplified to a simple term.

□

Theorem 2.23 *If any space contains an empty indicator, the value indicated in the space is the marked state. That is for any* $\boldsymbol{CI}$*-term* P, $P\boldsymbol{D}(\emptyset) = \boldsymbol{D}(\emptyset)$.

Proof Let M be any $\boldsymbol{CI}$-term containing an empty indicator. Then M is of the form $M \equiv P\boldsymbol{D}(\emptyset)$. By Theorem 2.22, a subterm P of M can be reduced to either of the following simple terms.

(1) $P \rightharpoondown \boldsymbol{D}(\emptyset)$

$$\begin{aligned} M \equiv P\boldsymbol{D}(\emptyset) &\rightharpoondown \boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset) \\ &\rightharpoondown \boldsymbol{D}(\emptyset) \end{aligned}$$

(2) $P \rightharpoondown \emptyset$

$$\begin{aligned} M \equiv P\boldsymbol{D}(\emptyset) &\rightharpoondown \emptyset\boldsymbol{D}(\emptyset) \\ &\rightharpoondown \boldsymbol{D}(\emptyset)\emptyset \\ &\rightharpoondown \boldsymbol{D}(\emptyset) \end{aligned}$$

Thereby, in each case the simplification of M can be reduced to a simple term $\boldsymbol{D}(\emptyset)$.
Hence, $M \equiv P\boldsymbol{D}(\emptyset) = \boldsymbol{D}(\emptyset)$. Therefore, M indicates the marked state.

□

Definition 2.24 (*i*) *Let M' stand for any number, greater than zero, of* $\boldsymbol{CI}$*-terms indicating the marked state. Call the value of M' a* dominant value.

(*ii*) *Let U' stand for any number of* $\boldsymbol{CI}$*-terms indicating the unmarked state. Call the value of U' a* recessive value.

(*iii*) *If any* $\boldsymbol{CI}$*-term M in a space s shows a* dominant value *in s, then the value of M is the* marked state. *Otherwise, the value of M is the* unmarked state. *(called Rule of dominance)*

By the above definition, we get the following equations:

(*i*) $M' = \boldsymbol{D}(\emptyset)$

(*ii*) $U' = \emptyset$

Proposition 2.25 *For M', U', the following equations hold.*

(1) $M'M' = M'$

(2) $U'U' = U'$

(3) $M'U' = M'$

(4) $\boldsymbol{D}(M') = U'$

(5) $\boldsymbol{D}(U') = M'$

Proof By the definition, Let M', U' be the followings:

$$M' \equiv m\boldsymbol{D}(\emptyset) \qquad (m \in N),$$
$$U' \equiv n\emptyset \qquad (n \in N)$$

We only show the case of (1), (3) and (4).

(1) $$\begin{aligned} M'M' \equiv m\boldsymbol{D}(\emptyset)m\boldsymbol{D}(\emptyset) \equiv 2m\boldsymbol{D}(\emptyset) \\ &\rightharpoondown (2m\text{-}1)\boldsymbol{D}(\emptyset) \\ &\quad \vdots \\ &\rightharpoondown m\boldsymbol{D}(\emptyset) \equiv M' \end{aligned}$$

(3) $$\begin{aligned} M'U' &\equiv m\boldsymbol{D}(\emptyset)n\emptyset \\ &\rightharpoondown m\boldsymbol{D}(\emptyset)(n\text{-}1)\emptyset \end{aligned}$$

$$\vdots$$

$$\rightharpoonup mD(\emptyset) \equiv M'$$

(4) $D(M') \rightharpoonup D(D(\emptyset))$

$$\rightharpoonup \emptyset$$

$$\rightharpoonup N'$$

□

Theorem 2.26 *The simplification of any* ***CI****-term is unique. That is to say, if any* ***CI****-term M simplifies to a simple term M_S, then M cannot simplify to a simple term other than M_S.*

Proof Let M be any ***CI***-term in space s_0. Then there exists a natural number n such that $n = dph(M)$. By the definition, the space s_n is the deepest space of M. Moreover, the indicators covering s_n are empty, and they are the only contents of s_{n-1}. Being empty, each indicator in s_{n-1} can be seen to indicate only the marked state.

Now make a mark M', U' on the outside of each indicator in M as the following procedure:

(1) Make a mark M' on the outside of each indicator in s_{n-1}. Then no value in s_{n-1} is changed, since

$$D(\emptyset)M' \rightharpoonup D(\emptyset)D(\emptyset)$$
$$\rightharpoonup D(\emptyset)$$

Therefore, the value of M is unchanged.

(2) Any indicator in s_{n-2} either is the followings:

(*i*) $lD(\emptyset)$ $(l \in N, l \geq 0)$

Mark it with M' so that the same considerations in (1) apply.

(*ii*) $nD(m(D(\emptyset)M'))$ $(m,n \in N, m,n \geq 1)$

Mark it with U'. Then no value in s_{n-2} is changed, since

$$D(m(D(\emptyset)M'))U' \rightharpoonup D(m(D(\emptyset)M')\emptyset$$
$$\rightharpoonup D(m(D(\emptyset)M'))$$

Therefore, the value of M is unchanged.

(3) Any indicator in s_{n-3} either is the followings:

(*i*) $lD(\emptyset)$ $(l \in N, l \geq 0)$

Mark it with M' so that the same considerations in (1) apply.

(*ii*) $kD(n(D(P)U')m(D(Q)M'))$ $(k,m,n \in N, k \geq 1, m \geq 1$ or $n \geq 1)$

If $m \geq 1$, mark it with U' so that the same consideration in (2)'s (ii) apply.

Also if $m = 0$, do the same as (i). Therefore, the value of M is unchanged.

The procedure is subsequent spaces to s_0 requires no additional consideration. Thus, by the procedure, each indicator in M is uniquely marked with M' or U'. Therefore, by the rule of dominance, a unique value of M in s_0 is determined. But the procedure leaves the value of M unchanged. Therefore, the simplification of any ***CI***-term is unique.

□

Corollary 2.27 *The complication of any simple term is unique. That is to say, the value of any* ***CI****-term constructed by taking steps from a given simple term is distinct from the value of any* ***CI****-term constructed from a different simple term.*

Definition 2.28 *A calculus that does not confuse a distinction it intends will be said to be consistent, where confuse a distinction is an equation of the from $M = D(M)$.*

Theorem 2.29 *Identical **CI**-terms express the same value. That is to say, in any **CI**-term M, M = M.*

Theorem 2.30 ***CI**-terms of the same value can be identified.*

Theorem 2.31 ***CI**-terms equivalent to an idential term are equivalent to one another. That is to say, in any **CI**-terms M, N, V, if M = V and N = V, then M = N.*

Theorem 2.32 *For any **CI**-terms M,* $\boldsymbol{D}(\boldsymbol{D}(M)M) = \varnothing$.

Proof By theorem 2.22, M can be reduced to either of the following simple terms:

(1) $M = \boldsymbol{D}(\varnothing)$

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{D}(M)M) &= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\varnothing))\boldsymbol{D}(\varnothing)) && (R1, R2)\\ &= \boldsymbol{D}(\varnothing \boldsymbol{D}(\varnothing)) && (A2)\\ &= \boldsymbol{D}(\boldsymbol{D}(\varnothing)\ \varnothing) && (A4)\\ &= \boldsymbol{D}(\boldsymbol{D}(\varnothing)) && (A6)\\ &= \varnothing && (A2) \end{aligned}$$

(2) $M = \varnothing$

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{D}(M)M) &= \boldsymbol{D}(\boldsymbol{D}(\varnothing)\ \varnothing) && (R1, R2)\\ &= \boldsymbol{D}(\boldsymbol{D}(\varnothing)) && (A6)\\ &= \varnothing && (A2) \end{aligned}$$

There is no other case of M, and there is no other way of substituting any case of M.

□

Theorem 2.33 *For any **CI**-terms M, N, L,* $\boldsymbol{D}(\boldsymbol{D}(ML)\boldsymbol{D}(NL)) = \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))L$.

Proof By theorem 2.22, L can be reduced to either of the following simple term:

(1) $L = \boldsymbol{D}(\varnothing)$

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{D}(ML)\boldsymbol{D}(NL)) &= \boldsymbol{D}(\boldsymbol{D}(M\boldsymbol{D}(\varnothing))\boldsymbol{D}(N\boldsymbol{D}(\varnothing))) && (R1)\\ &= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\varnothing))\boldsymbol{D}(\boldsymbol{D}(\varnothing))) && (\text{Th.1.30})\\ &= \boldsymbol{D}(\varnothing\varnothing) && (A2)\\ &= \boldsymbol{D}(\varnothing) && (A6) \end{aligned}$$

And

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))L &= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))\boldsymbol{D}(\varnothing) && (R1)\\ &= \boldsymbol{D}(\varnothing) && (\text{Th.1.30}) \end{aligned}$$

(2) $L = \varnothing$

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{D}(ML)\boldsymbol{D}(NL)) &= \boldsymbol{D}(\boldsymbol{D}(M\varnothing)\boldsymbol{D}(N\varnothing)) && (R1)\\ &= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N)) && (A6) \end{aligned}$$

And

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))L &= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))\varnothing && (R1)\\ &= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N)) && (A6) \end{aligned}$$

There is no other case of L, and there is no other way of substituting any case of L.

□

3 The primary algebra

3.1 The algebraic theory of I*-equality

Definition 3.1 (PA, the algebaric theory of I*-equality) (*i*) **PA**-*terms are the same as* **CI**-*terms.*

(*ii*) *The equations of* **PA** *are just* $M = N$, *for all* **PA**-*terms* M, N. *This algebaric theory is axiomatized by the following axioms and rules:*

(A1) $\boldsymbol{D}(\boldsymbol{D}(M)M) = \emptyset$ *(Position)*

(A2) $\boldsymbol{D}(\boldsymbol{D}(ML)\boldsymbol{D}(NL)) = \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))L$ *(Transposition)*

(A3) $M = M\emptyset$

(A4) $MN = NM$

(A5) $M(NL) = (MN)L$

(R1) *Substitution:*

If $E = F$ *and* $E \subset G$, *then infer* $G = [F/E]G$.

(R2) *Replacement:*

If $E = F$ *and any* **PA**-*term* G, *then infer* $[G/e]E = [G/e]F$ *where* e *is any variable occurred in* E *or* F.

(*iii*) *Derivability in* **PA** *of an equation is denoted by* $\mathbf{PA} \models M = N$ *or often just by* $M = N$.

(*iv*) *Call also this theory the* primary algebra.

Definition 3.2 (*i*) *Every* **PA**-*term has as values the letters* m *and* u - *standing for "marked" and "unmarked" states - and has as valuations mappings* $v : I \to \{m,u\}$ *such that*

(1) $v(e_i) \in \{m,u\}$, *(for* $i = 1,2,\ldots$*)*

(2) $v(\emptyset) = u$

(3) $v(\boldsymbol{D}(M)) = \begin{cases} m & \text{if } v(M) = u \\ u & \text{if } v(M) = m \end{cases}$

(4) $u(MN) = \begin{cases} m & \text{if either } v(M) = m \text{ or } v(N) = m \\ u & \text{otherwise} \end{cases}$

(*ii*) *An equation* $M = N$ *is valid in* **PA** (*i.e.* $\mathbf{PA} \models M = N$) *if* $v(M) = v(N)$ *for all such valuations* v.

Proposition 3.3 *Let* M,N,L,R,X,Y *be any* **PA**-*term. Then in any case, we have the following equations:*

(C1) $\boldsymbol{D}(\boldsymbol{D}(M)) = M$

(C2) $\boldsymbol{D}(MN)N = \boldsymbol{D}(M)N$

(C3) $\boldsymbol{D}(\emptyset)M = \boldsymbol{D}(\emptyset)$

(C4) $\boldsymbol{D}(\boldsymbol{D}(M)N)M = M$

(C5) $MM = M$

(C6) $\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))\boldsymbol{D}(\boldsymbol{D}(M)N) = M$

(C7) $\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M)N)L) = \boldsymbol{D}(ML)\boldsymbol{D}(\boldsymbol{D}(N)L)$

(C8) $\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(NR)\boldsymbol{D}(LR)) = \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N)\boldsymbol{D}(L))\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(R))$

(C9) $\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(R))\boldsymbol{D}(\boldsymbol{D}(N)\boldsymbol{D}R))\boldsymbol{D}(\boldsymbol{D}(X)R)\boldsymbol{D}(\boldsymbol{D}(Y)R))$
$= \boldsymbol{D}(\boldsymbol{D}(R)MN)\boldsymbol{D}(RXY)$

Proof Here we show three cases $C1$, $C7$ and $C8$ below. Others can also show in the same way.

$C1 : \boldsymbol{D}(\boldsymbol{D}(M))$

$= \emptyset\boldsymbol{D}(\boldsymbol{D}(M))$ (A3)

$= \boldsymbol{D}(\boldsymbol{D}(P)P)\boldsymbol{D}(\boldsymbol{D}(M))$ (A1,R2)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M))\boldsymbol{D}(M))\boldsymbol{D}(\boldsymbol{D}(M))$ ($P = \boldsymbol{D}(M)$, R1)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M))\boldsymbol{D}(M))\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M))M))$ (A2)

$= \boldsymbol{D}(\emptyset\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M))M))$ (A1)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M))M))$ (A3, R1)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M))M)\ \varnothing)$ (A3)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M))M\boldsymbol{D}(\boldsymbol{D}(M)M))$ (A1, R1)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M)))\boldsymbol{D}(\boldsymbol{D}(M)))M$ (A2, R2)

$= \varnothing M$ (A1)

$= M$ (A3)

$C7 : \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M)N)L)$

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M)))N)L)$ (C1)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(ML)\boldsymbol{D}(\boldsymbol{D}(N)L)))$ (A2)

$= \boldsymbol{D}(ML)\boldsymbol{D}(\boldsymbol{D}(N)L)$ (C1)

$C8 : \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(NR)\boldsymbol{D}(LR))$

$= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(NR)\boldsymbol{D}(LR))))$ (C1)

$= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(N)\boldsymbol{D}(L))R))$ (A2)

$= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N)\boldsymbol{D}(L))\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(R))$ (C7)

□

Theorem 3.4 *The scope of A2 can be extended to any number of divisions of the space s_{n+2}. That is in any case,*

$T1\ \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N)\cdots)R = \boldsymbol{D}(\boldsymbol{D}(MR)\boldsymbol{D}(NR)\cdots)$.

Proof We consider the cases in which s_{n+2} is divided into 0,1,2, and more than 2 divisions as the followings:

(1) case 0: $\boldsymbol{D}(\varnothing)R = \boldsymbol{D}(\varnothing)$ (C3)

(2) case 1: $\boldsymbol{D}(\boldsymbol{D}(M))R = MR$ (C1)

$= \boldsymbol{D}(\boldsymbol{D}(MR))$ (C1)

(3) case 2: $\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))R = \boldsymbol{D}(\boldsymbol{D}(MR)\boldsymbol{D}(NR))$ (A2)

(4) case more than 2:

$\boldsymbol{D}(\cdots\boldsymbol{D}(M)\boldsymbol{D}(N)\boldsymbol{D}(L))R$

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\cdots\boldsymbol{D}(M)))\boldsymbol{D}(N)))\boldsymbol{D}(L))R$ (C1)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\cdots\boldsymbol{D}(M)))\boldsymbol{D}(N))R)\boldsymbol{D}(LR))$ (A2)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\cdots\boldsymbol{D}(M\mathrm{R}))R)\boldsymbol{D}(NR)))\boldsymbol{D}(LR))$ (A2)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\cdots\boldsymbol{D}(M\mathrm{R})))\boldsymbol{D}(NR)))\boldsymbol{D}(LR))$ (A2)

$= \boldsymbol{D}(\cdots\boldsymbol{D}(MR)\boldsymbol{D}(NR)\boldsymbol{D}(LR))$ (C1)

□

Theorem 3.5 *The scope of C8 can be extended as in T1. That is in any case,*

$T2\ \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(NR)\boldsymbol{D}(LR)\cdots) = \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N)\boldsymbol{D}(L)\cdots)\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(R))$.

Proof $\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(NR)\boldsymbol{D}(LR)\cdots)$

$= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(NR)\boldsymbol{D}(LR)\cdots)))$ (C1)

$= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(N)\boldsymbol{D}(L)\cdots)R))$ (T1)

$= \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(N)\boldsymbol{D}(L)\cdots)R)\boldsymbol{D}(M))$ (A4)

$= \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N)\boldsymbol{D}(L)\cdots)\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(R))$ (C7)

Theorem 3.6 *The scope of C9 can be extended as in T1. That is in any case,*

$T3\ \boldsymbol{D}(\cdots\boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(R))\boldsymbol{D}(\boldsymbol{D}(N)\boldsymbol{D}(R))\boldsymbol{D}(\boldsymbol{D}(X)R)\boldsymbol{D}(\boldsymbol{D}(Y)R)\cdots)$

$= \boldsymbol{D}(\boldsymbol{D}(R)MN\cdots)\boldsymbol{D}(RXY\cdots)$.

Proof $D(\cdots D(D(M)D(R))D(D(N)D(R))D(D(X)R)D(D(Y)R\cdots)$

$$= D(\cdots D(D(M)D(R))D(D(N)D(R))D(D(D(D(X)R)D(D(Y)R)\cdots))) \quad (C1)$$
$$= D(\cdots D(D(M)D(R))D(D(N)D(R))D(D(D(D(X))D(D(Y))\cdots)R)) \quad (T1)$$
$$= D(\cdots D(D(M)D(R))D(D(N)D(R))D(D(XY\cdots)R)) \quad (C1)$$
$$= D(D(D(XY\cdots)R)\cdots D(D(M)D(R))D(D(N)D(R))\cdots) \quad (A4)$$
$$= D(D(D(XY\cdots)R)\cdots D(D(M))D(N))\cdots)D(D(D(XY\cdots)R)D(D(R))) \quad (T2)$$
$$= D(D(D(XY\cdots)R)MN\cdots)D(D(D(XY\cdots)R)R) \quad (C1)$$
$$= D(D(D(XY\cdots)R)MN\cdots)D(D(D(XY\cdots))R) \quad (C2)$$
$$= D(D(D(XY\cdots)R)MN\cdots)D(RXY\cdots) \quad (C1)$$
$$= D(MN\cdots D(D(XY\cdots)R))D(RXY\cdots) \quad (A4)$$
$$= D(MN\cdots D(D(XY\cdots)R)D(RXY\cdots))D(RXY\cdots) \quad (C2)$$
$$= D(MN\cdots D(D(D(R))D(XY\cdots))D(D(D(R))XY\cdots))D(RXY\cdots) \quad (C1)$$
$$= D(MN\cdots D(R))D(RXY\cdots) \quad (C6)$$
$$= D(D(R)MN\cdots)D(RXY\cdots) \quad (A4)$$

□

Theorem 3.7 *The generative process in C2 can be extended to any space not shallower than that in which the generated variable first appears.*

Proof We consider the cases in which a variable is generated in spaces 0,1, and more than 1 space deeper than the space of the variable of origin.

(1) case 0: $D(D(D(\cdots M)N)L)G = D(D(D(\cdots M)N)L)GG \quad (C5)$

(2) case 1: $D(D(D(\cdots M)N)L)G = D(D(D(\cdots M)N)LG)G \quad (C2)$

(3) case more than 1:

$$D(D(D(\cdots M)N)L)G$$
$$= D(D(D(\cdots M)N)LG)G \quad (C2)$$
$$= D(D(D(\cdots M)N)GL)G \quad (A4)$$
$$= D(D(D(\cdots M)NG)GL)G \quad (C2)$$
$$= D(D(D(\cdots M)NG)L)G \quad (C2)$$

And so on. It is plain that any space not shallower than that in which G stands can be reached.

□

Theorem 3.8 *From any given **PA**-term, an equivalent term not more than two indicators deep can be derived.*

Theorem 3.9 *From any given **PA**-term, an equivalent term can be derived so as to contain not more than two appearances of any given variable.*

Proof Let M be any ***PA***-term. If M has no variable, then the proof is trivial. So we may confine our consideration to the case of a variable e contained in M. Now by $C1$ and theorem 3.8,

$M = \cdots D(D(eN)Q)D(D(eM)P)FD(eX)D(eY)\cdots$, where $M, N, \ldots, P, Q, \ldots, X, Y, \ldots$ and F stand for subterms appropriate to the term M,

$$= \cdots D(D(D(D(e))D(D(N)))Q)D(D(eM)P)FD(eX)D(eY)\cdots \quad (C1)$$
$$= \cdots D(D(D(D(e)Q)D(D(N)Q)))D(D(eM)P)FD(eX)D(eY)\cdots \quad (A2)$$
$$= \cdots D(D(e)Q)D(D(N)Q)D(D(eM)P)FD(eX)D(eY)\cdots \quad (C1)$$
$$= \cdots D(D(e)Q)D(D(N)Q)D(D(e)P)D(D(M)P)FD(eX)D(eY)\cdots \quad (A2,C1)$$
$$= \cdots D(D(e)Q)D(D(e)P)FD(D(M)P)D(D(N)Q)\cdots D(eX)D(eY)\cdots \quad (A4)$$

$$= \cdots \boldsymbol{D}(\boldsymbol{D}(e)Q)\boldsymbol{D}(\boldsymbol{D}(e)P)G\boldsymbol{D}(eX)\boldsymbol{D}(eY)\cdots$$
where $G = F\boldsymbol{D}(\boldsymbol{D}(M)P)\boldsymbol{D}(\boldsymbol{D}(N)Q)\cdots$
$$= \boldsymbol{D}(\boldsymbol{D}(\cdots\boldsymbol{D}(\boldsymbol{D}(e)Q)\boldsymbol{D}(\boldsymbol{D}(e)P)))\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(eX)\boldsymbol{D}(eY)\cdots))G \quad (C1)$$
$$= \boldsymbol{D}(\boldsymbol{D}(\cdots\boldsymbol{D}(Q)\boldsymbol{D}(P))\boldsymbol{D}(e))\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(X)\boldsymbol{D}(Y)\cdots)e)G \quad (T1)$$

Definition 3.10 *Let a variable e in a space s_q oscillate between the limits of its value M',U'.*

(1) *If the value of every other indicator in s_q is U', the oscillation of e will be transmitted through s_q and seen as a variation in the value of the boundary of s_q to s_{q-1}. Under this condition call s_q* transparent.

(2) *If the value of any other indicator in s_q is M', nothing will be transmitted through s_q. Under this condition call s_q* opaque.

(3) *With regard to an oscillation in the value of a variable, the space outside the variable is either* transparent or opaque. *(Principle of transmission)*

Theorem 3.11 *If PA-terms are equivalent in every case of one variable, they are equivalent.*

3.2 Completeness and Independency

Theorem 3.12 (Completeness) *The primary algebra is complete. That is, M = N can be proved in the arithmetic if and only if M = N can be derived from the primary algebra.*

Proof Because of the rules of algebraic manipulation, it is immediate that if an equivalence $M = N$ is derivable from the axioms of primary algebra, then it is valid in the arithmetic.

Thus assume, conversely, that $M = N$ is a valid arithmetic formula. We show now that $M = N$ is derivable from the axioms of primary algebra. The proof proceeds by induction on the number n of variables contained in M, N.

(1) $n = 0$

In this case, $M = N$ contains no variable, and we need to show that if $M = N$ contains no variable, it is derivable in the algebra. We see in the proofs of theorem 2.22 - 2.26 and corollary 2.27 that all arithmetical equations are provable in the arithmetic. It remains to show that they are derivable in the algebra.

In $C3$ let $M = \boldsymbol{D}(\emptyset)$ to give $\boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset) = \boldsymbol{D}(\emptyset)$ and this is $A1$(Number).

In $C1$ let $M = \emptyset$ to give $\boldsymbol{D}(\boldsymbol{D}(\emptyset)) = \emptyset$ and this is $A2$(Order). Thus the axioms of the arithmetic are derivable in the primary algebra. and so if $M = N$ contains no variable it is derivable in the algebra.

(2) Assume that we have established the theorem for term containing less than n variables. Consider now terms M, N containing a total of n distinct variables. By theorem 3.8 and 3.9, we can reduce M, N to their canonical form with respect to a variable e:

$(E1)$ $M = \boldsymbol{D}(\boldsymbol{D}(e)A_1)\boldsymbol{D}(eA_2)A_3$,

$(E2)$ $N = \boldsymbol{D}(\boldsymbol{D}(e)B_1)\boldsymbol{D}(eB_2)B_3$,

since this reduction is proved with algebraic steps only. By hypothesis $\boldsymbol{CI} \vdash M = N$, we will get the following equation:

$(*)$ $\boldsymbol{D}(\boldsymbol{D}(e)A_1)\boldsymbol{D}(eA_2)A_3 = \boldsymbol{D}(\boldsymbol{D}(e)B_1)\boldsymbol{D}(eB_2)B_3$,

Our target is to show that (*) is derivable in the primary algebra. Thus by substitution we find that

$(E3)$ $\boldsymbol{D}(A_1)A_3 = \boldsymbol{D}(B_1)B_3$ *if* $e = \boldsymbol{D}(\emptyset)$,

$(E4)$ $\boldsymbol{D}(A_2)A_3 = \boldsymbol{D}(B_2)B_3$ *if* $e = \emptyset$,

to be arithmetically true. However, these two equations contain less than n variables, and thus they are, by hypothesis, derivable in the primary algebra. Then we have the following steps:

$$\mathrm{M} = \boldsymbol{D}(\boldsymbol{D}(e)A_1)\boldsymbol{D}(eA_2)A_3 \quad (E1)$$

$= D(D(D(e)D(A_1))D(eD(A_2)))A_3$ (C9)

$= D(D(D(e)D(A_1)A_3)D(eD(A_2)A_3))$ (A2)

$= D(D(D(e)D(B_1)B_3)D(eD(B_2)B_3))$ (E3,E4)

$= D(D(e)B_1)D(eB_2)B_3$ (A2,C9)

$= N$ (E2)

Thus $M = N$ with n variables is derivable from the axioms of the primary algebra if $M = N$ with less than n variables is derivable.

This completes the induction step and the proof.

□

Theorem 3.13 *The initials (Position and Transposition) of the primary algebra are independent. That is to say, given Position* (A1) *as the only initial, we cannot find Transposition* (A2) *as a consequence, and also given Transposition* (A2) *as the only initial, we cannot find Position* (A1) *as a consequence.*

4 The calculus interpreted for logic

4.1 The system PC of propositional calculus

Let $L_{PC} = \langle L_{PC}, \neg, \vee, \perp \rangle$ be the propositional language consisting of an infinite denumerable set of variables $p_1, p_2, p_3, \ldots$, a constant $\perp$ (false) and the truth functional connectives; $\neg$(negation) and $\vee$(disjunction). Then we have the following. Also see Schwartz's work [8] with reference to this section.

Definition 4.1 (*i*) *The set of* ***PC***-formulas *P is defined inductively as fallows:*

(1) *All propositional variables and a constant* $\perp$ *are* ***PC****-formulas (called atomic formulas).*

(2) *If P is a* ***PC****-formula, then* $\neg P$ *is a* ***PC****-formula.*

(3) *If P and Q are any* ***PC****-formulas, then* $(P \vee Q)$ *is a* ***PC****-formula.*

(4) *Nothing is a* ***PC****-formula except as required by (1), (2) and (3).*

(*ii*) *Call any* ***PC****-formulas with no variables* the closed ***PC***-formulas.

The further connectives may be introduced as mechanisms for abbreviation of complex formulas made up with $\neg$ and $\vee$: conjunction $P \wedge Q$, material implication $P \rightarrow Q$ and material equivalence $P \leftrightarrow Q$ are the abbreviation of $\neg(\neg P \vee \neg Q)$, $\neg P \vee Q$ and $(P \rightarrow Q) \wedge (Q \rightarrow P)$, respectively. And also the constant $\perp$ can be defined by $P \wedge \neg P$. Parenthesis are dropped when the intended grouping is clear, and note that $\neg$ has priority over $\vee$.

Definition 4.2 (PC,the system of classical propositional calculus)

(*i*) *This system is axiomatized by the following axioms and rules (originated with Hilbert and Ackerman):*

(A1) $(P \vee P) \rightarrow P$

(A2) $P \rightarrow (P \vee Q)$

(A3) $(P \vee Q) \rightarrow (Q \vee P)$

(A4) $(P \rightarrow Q) \rightarrow ((R \vee P) \rightarrow (R \vee Q))$

(A5) $\perp \leftrightarrow (P \wedge \neg P)$

(R1) *Modus Ponens:*

If P and $P \rightarrow Q$*, then infer Q.*

(R2) Uniform Substitution:

If P, then infer $P(Q/p_i)$*, where the latter denotes the formula that is obtained from P by replacing every* *occurrence of the variable p_i with an occurrence of the formula Q (and if p_i does not occur in P, then $P(Q/p_i)$ is just P).*

(*ii*) *Provability in* **PC** *of a theorem is denoted by* $\boldsymbol{PC} \vdash P$.

Definition 4.3 (*i*) *The semantics* $\Sigma(\boldsymbol{PC})$ *for* **PC** *has as* truth values *the numbers 1and 0, standing true and false respectively, and* truth valuations *are all the mappings* $V : P \to \{0,1\}$ *such that*

(1) $V(p_i) \in \{0,1\}$, (*for* $i = 1,2,\ldots..$)

(2) $V(\neg P) = 1 - V(P)$,

(3) $V(P \vee Q) = max(V(P), V(Q))$.

(*ii*) *A formula P is a* tautology *of* $\Sigma(\boldsymbol{PC})$ *if* $V(P) = 1$ *for all valuatoins V of* $\Sigma(\boldsymbol{PC})$. *(we specify* $\Sigma(\boldsymbol{PC}) \models P$*)*

Notes that truth valuations act on abbreviated formulas in the correct ways. For example we have $V(P \wedge Q) \Longleftrightarrow V(P) = 1$ and $V(Q) = 1$.

Theorem 4.4 (Completeness) *P is a tautology of* $\Sigma(\boldsymbol{PC})$ *if and only if* $\boldsymbol{PC} \vdash P$.

4.2 The systems CI and PC are Isomorphic

Definition 4.5 *A translation* γ *of* **CI** *into* **PC** *may now be defined by the followings:*

(1) $\gamma(e_i) = p_i$, (*for* $i = 1,2,\ldots$)

(2) $\gamma(\emptyset) = \perp$,

(3) $\gamma(\boldsymbol{D}(M)) = \neg\gamma(M)$,

(4) $\gamma(MN) = \gamma(M) \vee \gamma(N)$.

Proposition 4.6 *The mapping* γ *is well-defined and one-to-one. Hence, the inverse translation* γ^{-1} *is also well-defined and one-to-one.*

Theorem 4.7 ***CI*** *is isomorphic with* **PC**. *That is,* $\boldsymbol{CI} \vdash M = \boldsymbol{D}(\emptyset)$ *iff* $\boldsymbol{PC} \vdash \gamma(M)$ *for any* ***CI****-term M.*

Proof By completeness theorem of ***CI*** and **PC**, we get the following results for any ***CI***-term M:

$\boldsymbol{CI} \vdash M = \boldsymbol{D}(\emptyset)$ iff $\boldsymbol{PA} \models M = \boldsymbol{D}(\emptyset)$ (*Th*.3.12)

$\boldsymbol{PC} \vdash \gamma(M)$ iff $\Sigma(\boldsymbol{PC}) \models \gamma(M)$ (*Th*.4.4)

So, we need only to show the following proposition:

(*) $\boldsymbol{PA} \models M = \boldsymbol{D}(\emptyset)$ iff $\Sigma(\boldsymbol{PC}) \models \gamma(M)$

At first we observed that a one-to-one correspondence between valuations v of **PA** and truth valuations V of $\Sigma(\boldsymbol{PC})$ is given by $v(e_i) = m$ iff $V(p_i) = 1$ which implies that $v(e_i) = u$ iff $V(p_i) = 0$, since the respective valuation mappings are uniquely determined by their action on the e_i and p_i, ($i = 1,2,\ldots$). Hence it is sufficient to show: where V corresponds to v,

$v(M) = m$ iff $V(\gamma(M)) = 1$.

We show this by induction for the number of nesting indicators of a term. Suppose that M has depth n and that V corresponds to v.

(1) $n = 0$: there are two possibilities.

(*i*) $M \equiv e_i$ ($i = 1,2,\ldots$): then $\gamma(e_i) = p_i$, by definition of γ, so $v(M) = m$ iff $V(\gamma(M)) = 1$ by the correspondence of v and V.

(*ii*) $M \equiv \emptyset$: then $\gamma(M) = \bot$, and the case holds by default. There is no v such that $v(\emptyset) = m$, and there is no V such that $V(\bot) = 1$.

(2) $n > 0$: again there are two possibilities.

(*i*) $M \equiv \boldsymbol{D}(N)$: then N has depth less than n and the induction hypothesis yields:

$v(N) = m$ iff $V(\gamma(N)) = 1$.

This implies that

$$v(N) = u \text{ iff } V(\gamma(N)) = 0. \tag{1}$$

Thus,

$$\begin{aligned} v(M) = m \ &\text{iff } v(N) = u. && \text{(def. of } v\text{)} \\ &\text{iff } V(\gamma(N)) = 0. && \text{(by (1))} \\ &\text{iff } V(\neg\gamma(N)) = 1. && \text{(def. of } V\text{)} \\ &\text{iff } V(\gamma(M)) = 1. && \text{(def. of } \gamma\text{)} \end{aligned}$$

(*ii*) $M \equiv NL$: then each of N and L have depth less than n, and the induction hypothesis provides

$$v(N) = m \text{ iff } V(\gamma(N)) = 1. \tag{2}$$

$$v(L) = m \text{ iff } V(\gamma(L)) = 1. \tag{3}$$

Then

$$\begin{aligned} v(M) = m \ &\text{iff either } v(N) = m \text{ or } v(L) = m. && \text{(def. of } v\text{)} \\ &\text{iff either } V(\gamma(N)) = 1 \text{ or } V(\gamma(L)) = 1. && \text{(by (2),(3))} \\ &\text{iff } V(\gamma(N) \vee \gamma(L)) = 1. && \text{(def. of } V\text{)} \\ &\text{iff } V(\gamma(NL)) = 1. && \text{(def. of } \gamma\text{)} \end{aligned}$$

This completes the proof.

The following theorem shows that equality in ***CI*** is isomorphic with logical equivalence in ***PC***.

Theorem 4.8 *For all equations $M = N$ of **CI**, **CI** $\vdash M = N$ if and only if **PC** $\vdash \gamma(M) \leftrightarrow \gamma(N)$.*

Proof Since γ^{-1} is well-defined, theorem 4.7 yields

$\boldsymbol{PC} \vdash \gamma(M) \leftrightarrow \gamma(N)$ iff $\boldsymbol{CI} \vdash \gamma^{-1}(\gamma(M) \leftrightarrow \gamma(N)) = \boldsymbol{D}(\emptyset)$.

Thus, it is sufficient to show that

$\boldsymbol{CI} \vdash \gamma^{-1}(\gamma(M) \leftrightarrow \gamma(N)) = \boldsymbol{D}(\emptyset)$ iff $\boldsymbol{CI} \vdash M = N$.

But then, by the completeness of ***CI*** with respect to ***PA*** this makes it sufficient to show:

$\boldsymbol{PA} \models \gamma^{-1}(\gamma(M) \leftrightarrow \gamma(N)) = \boldsymbol{D}(\emptyset)$ iff $\boldsymbol{PA} \models M = N$.

By definition of γ, and the definition of the abbreviating connective $\leftrightarrow$, $\gamma^{-1}(\gamma(M) \leftrightarrow \gamma(N))$ is the term of ***CI*** $\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(M)N)\boldsymbol{D}(\boldsymbol{D}(N)M))$. (Denote this term by G)

Suppose $\boldsymbol{PA} \models G = \boldsymbol{D}(\emptyset)$. Then $v(G) = m$, for all v in ***PA***. By inspection of G, for any v, if $v(M) \neq v(N)$, then $V(G) \neq m$. Hence, for all v, $v(M) = v(N)$, and $\boldsymbol{PA} \models M = N$.

Next suppose $\boldsymbol{PA} \models M = N$. Then, for all v in ***PA***, either $v(M) = v(N) = m$ or $v(M) = v(N) = u$. By inspection of G, in either case $v(G) = m$. Thus, $\boldsymbol{PA} \models G = \boldsymbol{D}(\emptyset)$.

□

5 Conclusion

The calculus of indication proposed by G. Spencer-Brown may be regarded as an algebraic system consisting of two elements: a void space and a cross which reflects the operator-operand polarity, and two binary operations: juxtaposition and a kind of exponentiation. In order to show explicitly how these operations and thier polarity execute, at first, we defined a term reduction representation arised in the calculus of indication, and then introduced a formal theory ***CI***, corresponding to the primary arithmetic, based on I*-equality of ***CI***-terms. Moreover, we also introduced an algebric theory ***PA*** of I*-equality corresponding to the primary algebra in the same manner of his book [9], and then demonstrated an interpretation of ***PA*** within the classical propositional logic by guiding principle of appendix 2 in his book.

Several scholars ([1],[3],[7]) have already examined the relationship of the calculus of indication to other Boolean algebra with some distinct primitives. In [1], Banaschewski showed that the primary algebra may be isomorphically mapped into Boolean algebra with $\vee$(inclusive addition) and $\oplus$(exclusive addition) as primitives. We will review the result in our terminology as follows: Let $A_{PA} = \langle L_{CI}, \boldsymbol{D}, \emptyset \rangle (=L_{CI})$ be the primary algebra and $A_B = \langle L_A, \vee, \oplus, 0, 1 \rangle$ Boolean algebra. Now if we map the set $\{\emptyset, \boldsymbol{D}(\emptyset)\}$ into the set $\{0,1\}$, then we can define any juxtaposition xy in A_{PA} by an inclusive addition $x \vee y$ in A_B because it holds that $(A1)$ $\boldsymbol{D}(\emptyset)\boldsymbol{D}(\emptyset) = \boldsymbol{D}(\emptyset)$ and $(A6)$ $M = M\emptyset$ in definition 2.21 imply (1) $1 \vee 1 = 1$, (2) $1 \vee 0 = 1$, (3) $0 \vee 1 = 1$ and (4) $0 \vee 0 = 0$. Moreover, we can define any exponentiation $\boldsymbol{D}(x)$ in A_{PA} by an exclusive addition $x \oplus 1$ in A_B because that $(A2)$ $\boldsymbol{D}(\boldsymbol{D}(\emptyset)) = \emptyset$, $(A3)$ $M = M$ and $(A6)$ $M = M\emptyset$ in definition 2.21 imply (1) $1 \oplus 1 = 0$, (2) $1 \oplus 0 = 1$, (3) $0 \oplus 1 = 1$ and (4) $0 \oplus 0 = 0$. Hence the primary algebra ***PA*** can be viewed as Boolean algebra A_B. Conversely, if we map the set $\{0, 1, x \vee y, x \oplus y\}$ into the set $\{\emptyset, \boldsymbol{D}(\emptyset), xy, \boldsymbol{D}(\boldsymbol{D}(x)y)\boldsymbol{D}(x\boldsymbol{D}(y))\}$, then A_B can be viewed as ***PA***. Note that when we consider the indicational forms of Boolean equations, several notions condense into one, that is to say, the distinctor $\boldsymbol{D}()$ may have both a value 1 and an operator exponentiation $\oplus$. This condensation possess an advantage of computation in considering the indicational forms of Boolean equations. Furthermore, Kohout and Pinkava showed in [7] that the primary algebra ***PA*** also may be isomorphically mapped into the dual of Boolean algebra A_B, i.e., $A_B^{\circ} = \langle L_A, \wedge, \leftrightarrow, 0, 1 \rangle$ where $\wedge$(logical multiplication) is the operation dual to $\vee$ and $\leftrightarrow$ (logical equivalence) the operation dual to $\oplus$ with the following mapping: $\{\emptyset, \boldsymbol{D}(\emptyset), xy, \boldsymbol{D}(x)\} \mapsto \{1, 0, x \wedge y, x \leftrightarrow 0\}$. Hence we have observed that the primary algebra can handle by itself several types of Boolean algebra.

G. Spencer-Brown has also proposed re-entry forms in his treatment of the second order equations. Here one simple example of the re-entry form is a form f, that is identical with parts of its contents, i.e., $f = \phi(f)$ where ϕ is some indicational form containing f as a variable. Now if we consider the most simple reentrant form (1) $f = \boldsymbol{D}(f)$, then we have the following:

$$
\begin{aligned}
\emptyset &= \boldsymbol{D}(\boldsymbol{D}(p)p) && \text{(A1)}\\
&= \boldsymbol{D}(\boldsymbol{D}(f)f) && \text{(R2)}\\
&= \boldsymbol{D}(ff) && \text{(1)}\\
&= \boldsymbol{D}(f) && \text{(C5)}\\
&= f && \text{(1).}\\
&= \boldsymbol{D}(\emptyset). && \text{(1),(R1)}
\end{aligned}
$$

This equation: $\emptyset = \boldsymbol{D}(\emptyset)$ leads to a contradication in the primary algebra. In [11] and [12], Varela extends Brown's system to the consistant one by adding a third value, autonomous stste (we employ $\boldsymbol{A}(*)$ instead of the original self-cross), which represents a temporal oscillation of forms (also see [5]). In his calculus (called the extended calculus of indication) a simple re-entry form $f = \boldsymbol{D}(f)$ may view as the recursive action of $f \Rightarrow \boldsymbol{D}(f)$, thus we have:

$$\boldsymbol{D}(\emptyset) \Rightarrow \boldsymbol{D}(\boldsymbol{D}(\emptyset)) \Rightarrow \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\emptyset))) \Rightarrow \cdots,$$

and the autonomous state intends to a continuous oscillation of forms in time, that is to say, $f = \boldsymbol{D}(f) = \boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\boldsymbol{D}(\cdots)))) \equiv \boldsymbol{A}(*)$. Now if we define the set of ***ECI***-terms E as follows:

(1) All variables, a self-variable * and constant $\emptyset$ are ***ECI***-terms (called *atoms*)

(2) If M and N are any ***ECI***-terms, then (MN) is a ***ECI***-term (called a *calling*)

(3) If M is any ***ECI***-term, then $\boldsymbol{D}(M)$ is a ***ECI***-term (called a *distinction*)

(4) If M is any ***ECI***-term, then $\boldsymbol{A}(M)$ is a ***ECI***-term (called a *self-distinction*),

then the formal theory ***CI*** and its algebraic counterpart ***PA*** could be extended to fit Varela's calculus by axiomatizing the following, respectively:

$(A1)$ $\boldsymbol{D}(\emptyset)V = \boldsymbol{D}(\emptyset)$ where V is a marker(i.e, $\boldsymbol{D}(\emptyset)$, $\emptyset$, $\boldsymbol{A}(*)$) (Dominance)

$(A2)$ $\boldsymbol{D}^2(\emptyset) = \emptyset$ (Order)

$(A3)$ $\boldsymbol{D}(\boldsymbol{A}(*)) = \boldsymbol{A}(*)$ (Constancy)

$(A4)$ $2\boldsymbol{A}(*) = \boldsymbol{A}(*)$ (Number)

$(A1)$ $\boldsymbol{D}(\boldsymbol{D}(M)N)M = M$ (Occultation)

$(A2)$ $\boldsymbol{D}(\boldsymbol{D}(ML)\boldsymbol{D}(NL)) = \boldsymbol{D}(\boldsymbol{D}(M)\boldsymbol{D}(N))L$ (Transposition)

$(A3)$ $\boldsymbol{D}(M\boldsymbol{A}(*))M = M\boldsymbol{A}(*)$ (Autonomy) .

It was proved in [13] that Varela's extended calculus was a 3-valued extention of Brown's calculus. In [6], Orchard firstly pointed out the possibility that Brown's calculus can be viewed as one of non-Fregean system developed by Suszko [10]. The sentential calulus with identity (**SCI** for short) was proposed ny Suszko to realize some philosophical ideas of L. Wittgenstein's *Tractatus*. Here **SCI** is a classical two valued logic with an additional nontrivial connective identity $\equiv$ and it axioms, that is, $\equiv$ is not only an equivalence relation but also a congruence relation and at least as strong sa a material equivalence $\leftrightarrow$. So it holds that $(A \equiv B) \rightarrow (A \leftrightarrow B)$, but not the converse (called Fregean axiom). Since both calculi ***CI*** and **SCI** deal with some situations specified in a distinction $\boldsymbol{D}()$ or an identity $\equiv$, it would be of interest to know how to interpret each other and what modifications are needed in the interpretation.

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