A system of pair sentential calculus that has a representation of the Liar sentence

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Abstract

In this paper we will introduce a referential relation between pair-sentences similar to the identity connective $\equiv$ in **SCI**. Here pair-sentence statements of the form (A, B) read as "A is referential to B". Then by interpreting the pair-sentence (A, B) as a sequence of referential relation such that the referential recursive pattern $A\ B_0\ B_1\ B_2\ \ldots B_0\ B_1\ B_2\ \ldots$ holds, we will formalize a pair sentential calculus and represent the behavior of Liar sentence. To formalize the pair sentential calculus, we firstly introduce the stage numbers $i,\ j$ in which each of pair-sentence holds, i.e., (A^i, B^j) and which means a *situation* of A at a stage i is *referential* to the *situation* of B at a stage j. We also introduce a *referential cycle number* of (A^i, B^j) and by using of this cycle number, we may classify pair-sentences into two categories, that is, *categorical* and *paradoxical*. Then each Liar sentence has a referential cycle number of n such that $n \geq 2$ and is paradoxical.

Keywords: **SCI**, pair-sentence, Liar paradox, four-valued logic, paraconsistent logic.

1 Introduction

By the inspiration of L. Wittgenstein's *Tractatus* in which facts are constructed by *states of affairs (or situations)*, R. Suszko attempted to formalize an ontology of facts in *Tractatus* on the basis of Fregean scheme, and called it *non-Fregean logic* [12]. The sentential calculus with identity, **SCI** in short, is the most simplified version of his *non-Fregean logic* and obtained by adding the sentential identity connective $\equiv$ to the classical logic. Statements of the form $A \equiv B$ read as "A is identical with B" which means that the referent of two sentences are identical in the basis of Fregean scheme. In **SCI**, it is not assumed that all true (and similarily, all false) sentences have a common referent, called the *Fregean axiom*. Thus, we may think that **SCI** is addressing the area of many-valued logic, since sentences are allowed to have more than two values. But we must also be noticed that this is the referential many-valuedness, quite distinct from logical many-valuedness. If applying the theory of inference relation (finitary consequence operation) on the set of all formulas, then for any logic considered as an inference relation $\vdash$, we can find sets V of $0-1$ valued functions defined for all formulas, called *logical valuations* [13], that is, every logic may be regarded as logically two-valued. On the other hand, for the given formalized language $\mathcal{L}$, we may consider any algebraic structure $\mathcal{A}$ similar to $\mathcal{L}$ and maps of $\mathcal{L}$ to $\mathcal{A}$ satisfying some morphism conditions, called *algebraic valuations*. Then the formulas may have many algebraic values (admissible referents).

In this paper we will introduce a referential relation between pair-sentences similar to the identity connective $\equiv$ in **SCI**. Here pair-sentence statements of the form (A, B) read as "A is referential to B" which means that the referent of a sentence A is referential to the referent of a sentence B. When doing logical reasoning, it is usually assumed that several fundamental postulates implicitly hold by a priori. For example, the *principle of identity* says that "A is always A and not being $\neg A$", the *principle of contradiction* says that "A is not both A and

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$\neg A$", and the *principle of excluded middle* says that "either A is B or A is $\neg B$". Here if we do not admit these fundamental postulates, then several paradoxes appear in public and we could not proceed correctly to the formal reasoning. In order to overcome the matter, there exist several approaches to the problem from different point of views, that is, paraconsistent logic [2, 11], a theory of truth [10], naive semantics [5, 6] and self-reference/recursive forms [8, 9]. Here we will interpret pair-sentence (A, B) as a sequence of referential relation such that the referential recursive pattern $A\ B_0\ B_1\ B_2\ \ldots B_0\ B_1\ B_2\ \ldots$ holds by following the ideas of H. G. Herzberger and L. H. Kauffman. Then for the principle of identity sentence "A is A", we will get the pair-sentence (A, A) which satisfies a sequential form $A\ A\ A\ A\ A\ A\ldots$. Similarly, for a simple Liar sentence "this sentence is not true", we will get the pair-sentence $(A, \neg A)$ which satisfies a sequential form $A\ \neg A\ A\ \neg A\ A\ \neg A\ldots$. We have typically four referential relations, that is, $(\top, \top)$, $(\bot, \top)$, $(\top, \bot)$ and $(\bot, \bot)$. If we will axiomatize for a pair-sentence calculus similar to **SCI** manners, this system can be seen one of four-valued logic [1, 3], and could not deduce the principle of contradiction and excluded middle.

The logical axioms for **SCI** consist of two sets of schemata $\boldsymbol{TFA}$ (truth functional axioms) and $\boldsymbol{IDA}$ (identity axioms). Here $\boldsymbol{IDA}$ is the following:

(E1) $A \equiv A$

(E2) $(A \equiv B) \to (B \equiv A)$

(E3) $(A \equiv B) \land (B \equiv C) \to (A \equiv C)$

(C1) $(A \equiv B) \to (\neg A \equiv \neg B)$

(C2) $(A \equiv B) \land (C \equiv D) \to (A \land C) \equiv (B \land D)$

(C3) $(A \equiv B) \land (C \equiv D) \to (A \lor C) \equiv (B \lor D)$

(C4) $(A \equiv B) \land (C \equiv D) \to (A \to C) \equiv (B \to D)$

(C5) $(A \equiv B) \land (C \equiv D) \to (A \equiv C) \equiv (B \equiv D)$

(SI) $(A \equiv B) \to (A \to B)$

(E1)–(E3) and (C1)–(C5) show that the identity connective $\equiv$ is an equivalence and congruence relation respectively. From (SI) we get $A \leftrightarrow B \not\Rightarrow A \equiv B$ in general, which means **SCI** is non-Fregean logic. Every equation in the logical theorems of **SCI** is only a trivial (i.e., $A \equiv A$).

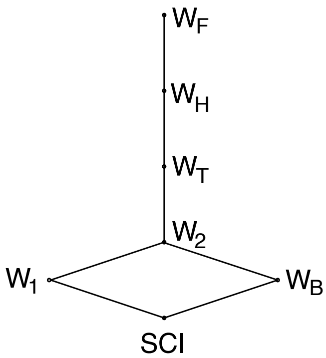

Figure 1: Relations between extensions of **SCI**

Moreover, many logics can be reconstructed as Suszko's theoreies of situation. For example, we define $\alpha \equiv \beta \iff \Box(\alpha \leftrightarrow \beta)$ in modal logic, then we have $\mathbf{W}_{\mathrm{T}}$ — **S4** and $\mathbf{W}_{\mathrm{H}}$ — **S5**. Also we define $\alpha \equiv \beta \iff (\alpha \Leftrightarrow \beta)$ in 3-valued Łukasiewicz logic where $\Leftrightarrow$ is a $Ł_3$ equivalence, then we have **SCI** — $\mathbf{Ł_3}$.

Here we consider a simple Liar sentence : "This sentence is not true". Of cource, **SCI** could not deal with the Liar sentence. Let's define A="This sentence is true", then $A \equiv \neg A$ because the referent of two sentences A and $\neg A$ are identical, but it's impossible logically by (SI). So, we will introduce a *referential relation* of pair-sentence similar to identity $\equiv$, i.e., $(A^i, \neg A^j)$: a *situation* of A at a stage i is *referential* to the *situation* of $\neg A$ at a stage j. In this paper, we will formalize a pair sentential calculus and represent the behavior of Liar sentence.

2 PSC Logic

In this paper we will propose a pair sentential calculus, **PSC** in short, which was obtained from the classical sentential calculus by adding a new pair-sentence constructor $(\_\,,\_)$ and its axioms of the form (A^i, B^j) which means "A at a stage i is referential to B at stage j" where i and j are referential stage numbers that A and B hold respectively.

2.1 Definitions

The formulas $\boldsymbol{FOR_P}$ of a language $\mathcal{L}_P$ of the sentential calculus with pair-sentence constructor are generated in the usual way from an infinite set Vp of sentential variables and constants $\top$ (true), $\bot$ (false) by the standard truth functional connectives $\neg$(negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (material implication) as well as the pair-sentence constructor $(\_\,,\_)$, i.e., $\mathcal{L}_P =< \boldsymbol{FOR_P}, \neg, \wedge, \vee, \rightarrow, (\_\,,\_), \top, \bot >$. Also we may use the same parentheses as auxiliary symbols even assume that the priority of each connective and constructor are weak as $\neg$, $\wedge$, $\vee$, $\rightarrow$, $(\_\,,\_)$ in order. Throughout this paper the letters p, q, r, p_1, ... will be used to denote any variables; the letters A, B, C, ... will denote formulas of a **PSC** language $\mathcal{L}_P$; the letters X, Y will denote sets of formulas. We will also use the following abbreviation: (i) $\top = (\top, \top)$, (ii) $\bot = (\bot, \bot)$, (iii) $\nabla = (\bot, \top)$, (iv) $\triangle = (\top, \bot)$ and (v) $A = (A, A)$. We will introduce several terminology with pair-sentence as the following.

Definition 2.1 (Pair-sentence)

(1) *If A is a subformula of B, then we say that pair-sentence (A^i, B^j) is a* circular referential relation*, where i and j are referential stage numbers that A and B hold respectively. Moreover, if A is not a subformula of B, then pair-sentence (A^i, B^j) is a* non-circular referential relation*.*

(2) *For a circular referential relation (A^i, B^j) such that the referential recursive pattern $A^i\ B_0^j\ B_1^{j+1}\ B_2^{j+2} \cdots B_0^{j+n} \cdots$ holds, the total referential stage numbers of B_0^j being recursively returned to itself is called a* referential cycle number *of (A^i, B^j) and the cycle number is n, $\mu(A^i, B^j) = n$ in symbol. On the other hand, for a non-circular referential relation (A^i, B^j) such that the referential recursive pattern $A^i\ B^j\ B^{j+1}\ B^{j+2} \cdots$ holds, the referential cycle number is always 1, i.e., $\mu(A^i, B^j) = 1$.*

(3) *If $\mu(A^i, B^j) = 1$ then we say that (A^i, B^j) is* categorical *and we will omit each referential stage number like (A, B). On the other hand, if $\mu(A^i, B^j) \geq 2$ then we say that (A^i, B^j) is* paradoxical*.*

(4) *The referential stage numbering of composed formulas is the following.*

- (i) $(\neg A)^i \iff \neg A^i$
- (ii) $(A \wedge B)^i \iff A^i \wedge B^i$
- (iii) $(A \vee B)^i \iff A^i \vee B^i$
- (iv) $(A \rightarrow B)^i \iff A^i \rightarrow B^i$

(v) $(A,B)^i \iff (A^i, B^i)$

The axiomatic system **PSC** for the language $\mathcal{L}_P$ is defined by the following way.

Definition 2.2 (PSC system) *The axiomatic system* **PSC** *consists of the two sets of schema* ***TFA*** *(truth functional axioms) and* ***PSA*** *(pair-sentence axioms) below. Furthermore* ***PSA*** *splits to the three sets of schema* ***EA*** *(equivalence axioms),* ***CA*** *(congruence axioms) and* ***PA*** *(pair-sentence axioms). The rule of inference is only modus ponens.*

$(A1)$ $A \to (B \to A)$

$(A2)$ $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$(A3)$ $A \wedge B \to A$

$(A4)$ $A \wedge B \to B$

$(A5)$ $A \to (B \to (A \wedge B))$

$(A6)$ $A \to A \vee B$

$(A7)$ $B \to A \vee B$

$(A8)$ $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

$(A9)$ $\neg A \to (A \to B)$

$(A10)$ $A \vee \neg A$

$(E1)$ (A, A)

$(E2)$ $(A^i, B^j) \to (B^j, A^i)$

$(E3)$ $(A^i, B^j) \wedge (B^k, C^l) \to (A^i, C^{l+(j-k)})$

$(C1)$ $(A^i, B^j) \to ((\neg A)^{i+1}, (\neg B)^{j+1})$

$(C2)$ $(A^i, B^j) \wedge (C^i, D^j) \to ((A \wedge C)^i, (B \wedge D)^j)$

$(C3)$ $(A^i, B^j) \wedge (C^i, D^j) \to ((A \vee C)^i, (B \vee D)^j)$

$(C4)$ $(A^i, B^j) \wedge (C^i, D^j) \to ((A \to C)^i, (B \to D)^j)$

$(C5)$ $(A^i, B^j) \wedge (C^k, D^l) \to ((A^i, C^k)^m, (B^j, D^l)^n)$

$(P1)$ $(A^i, B^j) \to (A^k \leftrightarrow B^{j+(k-i)})$

$(P2)$ $(A^i, B^k) \wedge (A^i, B^j) \to (A^{i+(k-j)}, B^k)$

$(P3)$ $A^i \to A^{i \pm n}$ *where* $n \geq 0$

$(P4)$ $A^i \to A^j$ *where* A *is related to other formulas only as the non-circular referential relation.*

(Mp) $\dfrac{A \quad A \to B}{B}$

The axioms in ***TFA*** with modus ponens as the single rule will give an axiomatic system ***CL*** for the classical sentential logic, and if we will restrict the pair-sentence formula (A^i, B^j) to a non-circular referential relation, then **PSC** is collapsed into pure **SCI** system because of regarding (A, B) as $A \equiv B$.

Definition 2.3 (Derivability) *Let* X *be a set of formulas in a language* $\mathcal{L}_P$*,* A *a formula and* **PSC** *a system in* $\mathcal{L}_P$*. Then we say that* A *is* derivable *from* X *in* **PSC***, we write* $\mathbf{PSC}, X \vdash A$ *iff there is a sequence of formulas* $B_1, \ldots, B_n (n \geq 0)$ *such that every formula in the sequence* $B_1, \ldots, B_n, A$ *is either a theorem of* **PSC***, or belongs to* X*, or is obtained by (Mp) rule from formulas occurring before it in the sequence, where if* $X = \emptyset$*, we write* $\mathbf{PSC} \vdash A$*, and we say that* A *is a* theorem *of* **PSC**.

2.2 Elementary Results

Theorem 2.4 *If we will restrict every pair-sentence formula* (A^i, B^j) *to the non-circular referential relation* (A, B)*, then* **PSC** *is collapsed into pure* **SCI** *system.*

Proof. Assume that all pair-sentences (A^i, B^j) are only the non-circular referential relation. Then we can omit each stage number because of its referential cycle number is 1 by Definition 2.1(2). In this case we can be identical (A, B) with $A \equiv B$ in **SCI** owing to the meaning of pair-sentence. Then axioms $\boldsymbol{EA}$ and $\boldsymbol{CA}$ show that the pair-sentence (A, B) satisfies an equivalence and congruence relation respectively. Moreover axiom (P1) show that $(A, B) \to (A \leftrightarrow B)$ holds. So all axioms need for **SCI** are satisfied by **PSC**.

□

Theorem 2.5 *While* **PSC** *is inconsistent as the ordinary logic, it is consistent as the pathological logic.*

Proof. If we will consider any circular referential relation (A^i, B^j) in **SCI**, then it is inconsistent, but consistent in **PSC** because each stage of pair-sentence is closed under classical in itself.

□

Theorem 2.6 *For any circular referential relation* (A^i, B^j) *where* $(i < j)$ *there exists a natural number* $n \geq 2$ *such that* $\mu(A^i, B^j) = n$ *[6].*

Theorem 2.7 *For any referential stage number* i, j, k *such that* $i < j < k$*, the following are logical theorems of* **PSC**.

(1) $(A^i, B^k) \wedge (A^i, B^j) \to (B^j, B^k)$

(2) $(A^i, B^k) \wedge (A^j, B^k) \to (A^i, A^j)$

(3) $(A^i, B^k) \wedge (A^j, B^k) \to (A^i, B^{i+(k-j)})$

(4) $(A^i, B^j) \wedge (B \to C)^j \to (A^i, C^j)$

(5) $(A^i, B^j) \wedge (A \to C)^i \to (C^i, B^j)$

Proof. (1) Suppose $(A^i, B^k) \wedge (A^i, B^j)$. Then we have $(A^i, B^k) \wedge (B^j, A^i)$ by (E2). So we get (B^j, B^k) by (E3). (2) is similar to (1). (3) Suppose $(A^i, B^k) \wedge (A^j, B^k)$. Then we have $(B^k, A^i) \wedge (B^k, A^j)$ by (E2) twice. So we get $(B^{i+(k-j)}, A^i)$ by (P2). Hence we get $(A^i, B^{i+(k-j)})$ by (E2). (4) Suppose $(A^i, B^j) \wedge (B \to C)^j$. Then we can deduce C at stage number j by (Mp). So we get (A^i, C^j). (5) is similar to (4).

□

Theorem 2.8 *For any pair-sentence form of* $(A^i, \neg A^j)$*, the referential cycle number of* $(A^i, \neg A^j)$ *is* $2(j - i)$.

Proof. Suppose $(A^i, \neg A^j)$. Then we have $(\neg A^{i+1}, \neg\neg A^{j+1})$, i.e., $(\neg A^{i+1}, A^{j+1})$ by (C1). So we get $(A^i, \neg A^j) \wedge (\neg A^{i+1}, A^{j+1}) \to (A^i, A^{2j-i})$ by (E3). Hence the referential cycle number is $\mu(A^i, \neg A^j) = (2j - i) - i = 2(j - i)$.

□

Theorem 2.9 *For axiom* (E2) $(A^i, B^j) \to (B^j, A^i)$*, whether both of* (A^i, B^j) *and* (B^j, A^i) *are circular referential relations or not is identical.*

Theorem 2.10 *Suppose that the pair-sentence form of* $(A^0, \neg A^1)$ *holds. Then* $\perp$ *is a logical theorem of* **PSC** *under the axiom* (P3) $A^i \to A^{i \pm n}$ *where* $n = 3$ *[4].*

Proof. Suppose $(A^0, \neg A^1)$.
(1) $[A^0]$ by Hypothesis
(2) $\neg A^1$ by (P1): $A^0 \wedge (A^0, \neg A^1) \to \neg A^1$
(3) $[\neg A^2]$ by Hypothesis
(4) A^1 by (3) and (P1): $\neg A^2 \wedge (A^0, \neg A^1) \to A^1$
(5) $(\perp)^1$ by (2) and (4)
(6) $(\perp)^2$ by (P4)
(7) A^2 by (3) and (6)
(8) $\neg A^3$ by (7) and (P1): $A^2 \wedge (A^0, \neg A^1) \to \neg A^3$
(9) $\neg A^0$ by (P3): $\neg A^3 \to \neg A^0$
(10) $\neg A^0$ by (1) and (9)
(11) A^1 by (C1), (10) and (P1): $\neg A^0 \wedge (\neg A^1, A^2) \to A^1$
(12) $\neg A^2$ by (11) and (P1): $A^1 \wedge (A^0, \neg A^1) \to A^2$
(13) $\neg A^3$ by (10) and (P3): $\neg A^0 \to \neg A^3$
(14) A^2 by (13) and (P1): $\neg A^3 \wedge (A^0, \neg A^1) \to A^2$
(15) $(\perp)^2$ by (12) and (14)
(16) $(\perp)^0$ by (15) and (P4)

□

3 Examples

We will investigate several circular referential relations.

Example 3.1 (A simple liar sentence)

> *"This sentence is not true".*

Remark. Now we define $A =$ "This sentence is true". Then a simple liar sentence is expressed by the pair-sentence formula $(A, \neg A)$. Suppose $(A^0, \neg A^1)$. Then we get the referential cycle number of $(A^0, \neg A^1)$ is 2 by Theorem 2.7.

Moreover, because of $\mu(A^0, \neg A^1) \geq 2$ this sentence is paradoxical by Definition 2.1(3).

□

Example 3.2 (Dialogue for Socrates and Plato 1)

> *Socrates : "My remarks are not true".*
>
> *Plato : "Socrates's remarks are true".*

Remark. Now we define $S =$ "Socrates's remarks are true" and $P =$ "Plato's remarks are true". Then we get the two pair-sentence formulas $(S, \neg S)$ and (P, S). Suppose $(S^0, \neg S^1)$ and (P^0, S^1). Then we get the referential cycle number of $(S^0, \neg S^1)$ is 2 by Theorem 2.7.

Moreover, the referential cycle number of P is also 2 as follows.
(1) $(P^0, \neg S^2)$ by (E3): $(P^0, S^1) \wedge (S^0, \neg S^1) \to (P^0, \neg S^2)$
(2) (P^0, S^3) by (1), (C1) and (E3): $(P^0, \neg S^2) \wedge (\neg S^1, S^2) \to (P^0, S^3)$
(3) (P^2, S^3) by (2) and (P2): $(P^0, S^3) \wedge (P^0, S^1) \to (P^2, S^3)$
(4) (P^0, P^2) by (3) and Theorem2.5(2): $(P^0, S^3) \wedge (P^2, S^3) \to (P^0, P^2)$

So two sentences are both paradoxical by Definition 2.1(3).

□

Example 3.3 (Dialogue for Socrates and Plato 2)

> *Socrates : "Plato's remarks are not true".*
>
> *Plato : "Socrates's remarks are true".*

Remark. Now we define S = "Socrates's remarks are true" and P = "Plato's remarks are true". Then we get the two pair-sentence formulas $(S, \neg P)$ and (P, S). Suppose $(S^0, \neg P^1)$ and (P^0, S^1). Then we get the referential cycle number of S and P as follows.
(1) $(\neg P^1, \neg S^2)$ by Hypothesis and (C1): $(P^0, S^1) \to (\neg P^1, \neg S^2)$
(2) $(S^0, \neg S^2)$ by (1) and (E3): $(S^0, \neg P^1) \wedge (\neg P^1, \neg S^2) \to (S^0, \neg S^2)$

So we get the referential cycle number of S is $\mu(S^0, \neg S^2) = 2(2-0) = 4$ by Theorem 2.7.

Similarly, the referential cycle number of P is 4 as follows.
(3) $(P^0, \neg P^2)$ by (E3): $(P^0, S^1) \wedge (S^0, \neg P^1) \to (P^0, \neg P^2)$

So two sentences are both paradoxical by Definition 2.1(3).

Next we will consider the sentence $S \wedge P$ means that "Both remarks of Socrates and Plato are true". Then we have the following calculation.
(4) $((S \wedge P)^0, (\neg P \wedge S)^1)$ by (C2)
(5) $(\neg P^1, \neg S^2)$ by (C1)
(6) $((\neg P \wedge S)^0, (\neg P \wedge \neg S)^1)$ by (5) and (C2)
(7) $((S \wedge P)^0, (\neg P \wedge \neg S)^2)$ by (4), (6) and (E3)
(8) $(\neg S^1, P^2)$ by (C1)
(9) $((\neg P \wedge \neg S)^0, (\neg S \wedge P)^1)$ by (8) and (C2)
(10) $((S \wedge P)^0, (\neg S \wedge P)^3)$ by (7), (9) and (E3)
(11) $((\neg S \wedge P)^0, (P \wedge S)^1)$ by (C2)
(12) $((S \wedge P)^0, (P \wedge S)^4)$ by (10), (11) and (E3)

So the referential cycle number of $S \wedge P$ is 4 and in this case it is also paradoxical.

□

Example 3.4 (Dialogue for Socrates and Plato 3)

Socrates : "Plato's remarks are not true".

Plato : "Socrates's remarks are not true".

Remark. Now we define S = "Socrates's remarks are true" and P = "Plato's remarks are true". Then we get the two pair-sentence formulas $(S, \neg P)$ and $(P, \neg S)$. Suppose $(S^0, \neg P^1)$ and $(P^0, \neg S^1)$. Then we get the referential cycle number of S is 2 as follows.
(1) $(\neg P^1, S^2)$ by Hypothesis and (C1)
(2) (S^0, S^2) by (1) and (E3): $(S^0, \neg P^1) \wedge (\neg P^1, S^2) \to (S^0, S^2)$

Also the referential cycle number of P is same as S. So two sentences are both paradoxical by Definition 2.1(3).

But if we consider the sentences of $S \wedge \neg P$ or $\neg S \wedge P$, then both cases are categorical. For example,
(3) $((S \wedge \neg P)^0, (\neg P \wedge S)^1)$ by (1) and (C2)
Hence we get $\mu(S \wedge \neg P) = 1$ by Theorem 2.7 and categorical.

□

Example 3.5 (Dialogue for Socrates, Plato and Aristoteles 1)

Socrates : "Plato's remarks are not true".

Plato : "Aristoteles's remarks are not true".

Aristoteles : "Socrates' remarks are true".

Remark. Now we define S = "Socrates's remarks are true", P = "Plato's remarks are true" and A = "Aristoteles's remarks are true". Then we get the three pair-sentence formulas $(S, \neg P)$, $(P, \neg A)$ and (A, S). Suppose $(S^0, \neg P^1)$, $(P^0, \neg A^1)$ and (A^0, S^1). Then all of the sentences $S \wedge P \wedge A$, $S \wedge P \wedge \neg A$, $S \wedge \neg P \wedge \neg A$, $\neg S \wedge P \wedge A$, $\neg S \wedge \neg P \wedge A$ and $\neg S \wedge \neg P \wedge \neg A$ have 3 as the referential cycle number and paradoxical. For example,
(1) $((S \wedge P \wedge A)^0, (\neg P \wedge \neg A \wedge S)^1)$ by Hypothesis and (C2)

(2) $((\neg P \wedge \neg A \wedge S)^0, (A \wedge \neg S \wedge \neg P)^1)$ by Hypothesis, (C1) and (C2)
(3) $(((S \wedge P \wedge A)^0, (A \wedge \neg S \wedge \neg P)^2)$ by (1), (2) and (E3)
(4) $((A \wedge \neg S \wedge \neg P)^0, (S \wedge P \wedge A)^1)$ by Hypothesis, (C1) and (C2)
(5) $((S \wedge P \wedge A)^0, (S \wedge P \wedge A)^3)$ by (3), (4) and (E3)
On the other hand, each of the sentences $S \wedge \neg P \wedge A$ and $\neg S \wedge P \wedge \neg A$ is categorical. For example,
(6) $((S \wedge \neg P \wedge A)^0, (\neg P \wedge A \wedge S)^1)$ by Hypothesis and (C2)

□

Example 3.6 (Dialogue for Socrates, Plato and Aristoteles 2)

Socrates : "Plato's remarks are not true".

Plato : "Aristoteles's remarks are not true".

Aristoteles : "Socrates' remarks are not true".

Remark. Now we define $S =$ "Socrates's remarks are true", $P =$ "Plato's remarks are true" and $A =$ "Aristoteles's remarks are true". Then we get the three pair-sentence formulas $(S, \neg P)$, $(P, \neg A)$ and $(A, \neg S)$. Suppose $(S^0, \neg P^1)$, $(P^0, \neg A^1)$ and $(A^0, \neg S^1)$. Then all of the sentences $S \wedge P \wedge \neg A$, $S \wedge \neg P \wedge A$, $S \wedge \neg P \wedge \neg A$, $\neg S \wedge P \wedge A$, $\neg S \wedge P \wedge \neg A$ and $\neg S \wedge \neg P \wedge A$
have 6 as the referential cycle number and paradoxical. For example,
(1) $((S \wedge P \wedge \neg A)^0, (\neg P \wedge \neg A \wedge S)^1)$ by Hypothesis and (C2)
(2) $((\neg P \wedge \neg A \wedge S)^0, (A \wedge S \wedge \neg P)^1)$ by Hypothesis, (C1) and (C2)
(3) $(((S \wedge P \wedge \neg A)^0, (A \wedge S \wedge \neg P)^2)$ by (1), (2) and (E3)
(4) $((A \wedge S \wedge \neg P)^0, (\neg S \wedge \neg P \wedge A)^1)$ by Hypothesis, (C1) and (C2)
(5) $((S \wedge P \wedge \neg A)^0, (\neg S \wedge \neg P \wedge A)^3)$ by (3), (4) and (E3)
(6) $((\neg S \wedge \neg P \wedge A)^0, (P \wedge A \wedge \neg S)^1)$ by Hypothesis, (C1) and (C2)
(7) $((S \wedge P \wedge \neg A)^0, (P \wedge A \wedge \neg S)^4)$ by (5), (6) and (E3)
(8) $((P \wedge A \wedge \neg S)^0, (\neg A \wedge \neg S \wedge P)^1)$ by Hypothesis, (C1) and (C2)
(9) $((S \wedge P \wedge \neg A)^0, (\neg A \wedge \neg S \wedge P)^5)$ by (7), (8) and (E3)
(10) $((\neg A \wedge \neg S \wedge P)^0, (S \wedge P \wedge \neg A)^1)$ by Hypothesis, (C1) and (C2)
(11) $((S \wedge P \wedge \neg A)^0, (S \wedge P \wedge \neg A)^6)$ by (9), (10) and (E3)
On the other hand, each of the sentences $S \wedge P \wedge A$ and $\neg S \wedge \neg P \wedge \neg A$ has 2 as the referential cycle number and also paradoxical. For example,
(12) $((S \wedge P \wedge A)^0, (\neg P \wedge \neg A \wedge \neg S)^1)$ by Hypothesis and (C2)
(13) $((\neg P \wedge \neg A \wedge \neg S)^0, (S \wedge P \wedge A)^1)$ by Hypothesis and (C2)
(14) $((S \wedge P \wedge A)^0, (S \wedge P \wedge A)^2)$ by (12), (13) and (E3)

□

Example 3.7 $(G, (F \vee (H \wedge \neg G)))$ *[4]*

Remark.
(1) $(G^0, (F \vee (H \wedge \neg G))^1)$ Hypothesis
(2) (F^0, F^1) Hypothesis
(3) (H^0, H^1) Hypothesis
(4) $(\neg G^1, \neg(F \vee (H \wedge \neg G))^2)$ by (1) and (C1)
where $\neg(F \vee (H \wedge \neg G)) = (\neg F \wedge (\neg H \vee G))$.
(5) $((H \wedge \neg G)^0, (H \wedge \neg F \wedge (\neg H \vee G))^1)$ by (3), (4) and (C2)
where (C2): $(H^0, H^1) \wedge (\neg G^0, \neg(F \vee (H \wedge \neg G))^1) \rightarrow$
$((H \wedge \neg G)^0, (H \wedge \neg F \wedge (\neg H \vee G))^1)$
(6) $((F \vee (H \wedge \neg G))^0, (F \vee (H \wedge G))^1)$ by (2), (5) and (C3)
(7) $(G^0, (F \vee (H \wedge G))^2)$ by (1), (6) and (E3)
(8) $((F \vee (H \wedge G))^0, (F \vee (H \wedge \neg G))^1)$ by similar to (4) – (6)

(9) $(G^0, (F \vee (H \wedge \neg G))^3)$ by (7), (8) and (E3)
Hence we get $\mu(G^0, (F \vee (H \wedge \neg G))^1) = 2$ and paradoxical.
Moreover $F \to G$ is provable as follows.
(10) $[F^0]$ by Hypothesis
(11) F^1 by (2), (10) and (P1): $F^0 \wedge (F^0, F^1) \to F^1$
(12) $(F \vee (H \wedge \neg G))^1$ by (11) and ($\vee$-introduction) in a stage 1
(13) G^0 by (12) and
(P1): $(F \vee (H \wedge \neg G))^1 \wedge (G^0, (F \vee (H \wedge \neg G))^1) \to G^0$
(14) $F^0 \to G^0$ by (10), (13) and ($\to$-introduction) in a stage 0
(15) $(F \to G)^0$

□

Example 3.8 $(G, (F \wedge H) \vee (F \wedge \neg H \wedge G) \vee (\neg F \wedge H \neg G))$ *[4]*

Remark.
(1) $(G^0, ((F \wedge H) \vee (F \wedge \neg H \wedge G) \vee (\neg F \wedge H \wedge \neg G))^1)$ Hypothesis
(2) (F^0, F^1) Hypothesis
(3) (H^0, H^1) Hypothesis
(4) Suppose $(F \wedge \neg H, \top)^1$, then we get $(F \wedge H, \bot)^1$, $(F \wedge \neg H \wedge G, G)^1$ and $(\neg F \wedge H \wedge \neg G, \bot)^1$. So, (G^0, G^1) and categorical.
(5) Suppose $(\neg F \wedge H, \top)^1$, then we get $(F \wedge H, \bot)^1$, $(F \wedge \neg H \wedge G, \bot)^1$ and $(\neg F \wedge H \wedge \neg G, \neg G)^1$. So, $(G^0, \neg G^1)$ and paradoxical.
(6) Suppose $(F \wedge H, \top)^1$, then we get $(F \wedge H, \top)^1$, $(F \wedge \neg H \wedge G, \bot)^1$ and $(\neg F \wedge H \wedge \neg G, \bot)^1$. So, $(G^0, \top^1)$ and categorical with truth.
(7) Suppose $(\neg F \wedge \neg H, \top)^1$, then we get $(F \wedge H, \bot)^1$, $(F \wedge \neg H \wedge G, \bot)^1$ and $(\neg F \wedge H \wedge \neg G, \bot)^1$. So, $(G^0, \bot^1)$ and categorical with false.

□

4 Conclusion

We proposed a pair sentential calculus **PSC** by extending **SCI** to deal with (A^i, B^j), which means "A at a stage i is referential to B at stage j" where i and j are referential stage numbers that A and B hold respectively. Here **PSC** is a conservative extension of **SCI** and can deal with the paradoxical sentences. While **PSC** is inconsistent as the ordinary logic, it is consistent as the pathological logic, so it is one of paraconsistent logic.
- further works

- some elementary extensions of **PSC**
- relations with other paraconsistent logics
- adequate algebraic semantics

References

[1] N. D. Belnap, Jr, A useful four-valued logic, In J. M. Dunn and G. Epstein (eds), *Modern Uses of Multiple-Valued Logic*, D. Reidel, 1977.

[2] N. da Costa, On the Theory of Inconsistent Formal Systems, *Notre Dame Journal of Formal Logic*,vol.15(4) (1974), pp.497–510.

[3] M. C. Fitting, kleene's logic, generalized, *Journal of Logic and Computation 1*, 1992, pp.797–810.

[4] A. Gupta and N. Belnap, *The Revision Theory of Truth*, MIT Press, Cambridge, 1993.

[5] H. G. Herzberger, Notes on naive semantics, *Journal of Philosophical Logic 11*, 1982, pp.61–102.

[6] H. G. Herzberger, Naive semantics and the Liar paradox, *Journal of Philosophy 79*, 1982, pp.479–497.

[7] T. Ishii, SCI for Pair-Sentence, *Short papers of the 13th Studia Logica International Conference on Trends in Logic XIII*, University of Łódź, 2014.

[8] L. H. Kauffman, De Morgan algebras — completeness and recursion, *Proceedings of the Eighth International Symposium on Multiple-Valued Logic*,(1978), pp.82–86.

[9] L. H. Kauffman, Complex number and algebraic logic, *Proceedings of the 10th International Symposium on Multiple-Valued Logic*,(1980), pp.82–86.

[10] S. Kripke, Outline of a theory of truth *Journal of Philosophy 72*, 1975, pp.690–716.

[11] G. Priest, Paraconsistent logic, In D. M. Gabbay and F. Guenthncr (eds), *Handbook of philosophical logic*, Kluwer academic, 2002.

[12] R. Suszko, Identity connective and modality, *Studia Logica*,vol.27(1971), pp.9–39.

[13] R. Suszko, The Fregean axiom and Polish mathematical logic in the 1920s, *Studia Logica*, vol.36(1977), pp.377–380.

[14] R. Wójcicki, R. Suszko's situational semantics, *Studia Logica*, vol.4(1984), pp.323–340.