A Class of Feature Spaces and Violation of the Triangle Inequality

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Abstract

A class of feature spaces that can violate the triangle inequality is proposed. The theory states that arbitrary N samples, in which all the distances between any pair of samples are given but their coordinates are unknown, can always assign the coordinates in the defined feature space after eigenvalue decomposition.

Keywords

Feature space, Metric space, Inner product, Triangle inequality

1. Introduction

The feature space is one of the basic concepts in information science including pattern recognition [1], multivariate analysis [2], optimization problems [3], and artificial neural networks [4]. Pattern recognition classifies input vectors in a feature space and a metric is critical that defines a norm between two vectors in the feature space. Multivariate analysis handles multidimensional vectors, which are supposed to be coordinates in a feature space. Optimization problems frequently require minimizing or maximizing vectors in a feature space. Artificial neural networks, one of the most successful mathematical models in information science, encode information of features in input and output vectors based on machine learning. An important characteristic of the feature space is quantitative calculation of an abstract human concept by using a real value metric. Another character is the utilization of linear algebra, which has many well-known mathematical techniques. It makes the feature space a popular tool in the science and engineering fields.

The concept of metric in vector spaces may be formally generalized in several manners. A pseudometric space is a space in which the distance between a pair of different vectors can be zero but the triangle inequality holds. Oppositely, a semimetric space is a space in which the distance between a pair of vectors is zero if and only if the vectors are identical but the triangle inequality can be violated among three elements. Similar metrics are utilized for formalizing various abstract concepts in mathematics such as topology or geometry.

The triangle inequality always holds on a usual feature space that has a metric because a metric acquires a defined distance. A normal metric feature space thus cannot process data violating the triangle inequality among three of vectors inside a feature space. This violation is sometimes appeared if a distance matrix among all the pairs of vectors in a feature space is given in which explicit coordinates of the vectors are unknown. Such situations occur in many cases in real problems, e.g., when one would like to classify relationships among many samples in a multi-dimensional feature space in which a sample is supposed to be characterized by multi-dimensional variables and only representative distances between any pair of samples are

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experimentally determined. If those data have violations of the triangle inequality, one would meet difficulty to resolve allocations of the data within a Euclidean space. One might thus conclude that the semimetric space is adequate because semimetric may violate triangle inequality. Although there are some applications of semimetric in engineering fields, mathematical theories should be expected to extend.

Here we describe the theory for a class of *N*-dimensional complex feature spaces and show the violation of triangle inequality can be logically existed without any conflict in that spaces. This theory is simple and expected to be applicable to methods such as pattern recognition or neural networks.

2. Definition of the proposed three feature spaces

A series of metric spaces in this study is defined by using *N*-dimensional complex vector spaces, C^N, that have new inner products, norms, and distances introduced this section.

DEFINITION 1 The three inner products are defined as,

$$(1) \qquad \langle v,w \rangle = \frac{{}^t vw + \overline{{}^t v}\,\overline{w}}{2} = \sum_{k=1}^{N} \frac{v_k w_k + \overline{v}_k\,\overline{w}_k}{2},$$

$$(2) \qquad \{v,w\} = \frac{{}^t vw - \overline{{}^t v}\,\overline{w}}{2i} = \sum_{k=1}^{N} \frac{v_k w_k - \overline{v}_k\,\overline{w}_k}{2i},$$

$$(3) \qquad [v,w] = \langle v,w \rangle + i\{v,w\} = {}^t vw = \sum_{k=1}^{N} v_k w_k,$$

where $v = (v_1, v_2, \cdots, v_N) \in C^N$, $w = (w_1, w_2, \cdots, w_N) \in C^N$, C^N is a *N*-dimensional complex vector space, ${}^t v$ is a transposed vector of v, and $\overline{w}$ is a complex conjugate of w.

Each inner product satisfies the following conditions, which are common to other ordinary inner products.

PROPOSITION 1   The inner product $\langle v,w \rangle$ satisfies

$$(4) \qquad \langle v,w \rangle = \langle w,v \rangle,$$

$$(5) \qquad \langle \alpha u + \beta v, w \rangle = \alpha \langle u,w \rangle + \beta \langle v,w \rangle,$$

$$(6) \qquad \langle v,v \rangle \in R,$$

where $u,v,w \in C^N$, and $\alpha, \beta \in R$.

PROOF.

(4) $\langle v, w \rangle = \sum_{k=1}^{N} \dfrac{v_k w_k + \overline{v_k}\,\overline{w_k}}{2} = \sum_{k=1}^{N} \dfrac{w_k v_k + \overline{w_k}\,\overline{v_k}}{2} = \langle w, v \rangle .$

(5) $\langle \alpha u + \beta v, w \rangle = \sum_{k=1}^{N} \dfrac{(\alpha u_k + \beta v_k) w_k + \overline{(\alpha u_k + \beta v_k)}\,\overline{w_k}}{2}$

$= \alpha \sum_{k=1}^{N} \dfrac{u_k w_k + \overline{u_k}\,\overline{w_k}}{2} + \beta \sum_{k=1}^{N} \dfrac{v_k w_k + \overline{v_k}\,\overline{w_k}}{2} = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$

(6) $\langle v, v \rangle = \sum_{k=1}^{N} \dfrac{v_k^{\,2} + \overline{v_k}^{\,2}}{2} = \sum_{k=1}^{N} \mathrm{Re}\!\left(v_k^{\,2}\right) \in R .$

PROPOSITION 2　　　　The inner product $\{v, w\}$ satisfies

(7)　　　$\{v, w\} = \{w, v\},$

(8)　　　$\{\alpha u + \beta v, w\} = \alpha\{u, w\} + \beta\{v, w\},$

(9)　　　$\{v, v\} \in R,$

where $u, v, w \in C^{N}$, and $\alpha, \beta \in R$.

PROOF.

(7) $\{v, w\} = \sum_{k=1}^{N} \dfrac{v_k w_k - \overline{v_k}\,\overline{w_k}}{2} = \sum_{k=1}^{N} \dfrac{w_k v_k - \overline{w_k}\,\overline{v_k}}{2} = \{w, v\} .$

(8) $\{\alpha u + \beta v, w\} = \sum_{k=1}^{N} \dfrac{(\alpha u_k + \beta v_k) w_k - \overline{(\alpha u_k + \beta v_k)}\,\overline{w_k}}{2i}$

$= \alpha \sum_{k=1}^{N} \dfrac{u_k w_k - \overline{u_k}\,\overline{w_k}}{2i} + \beta \sum_{k=1}^{N} \dfrac{v_k w_k - \overline{v_k}\,\overline{w_k}}{2i} = \alpha\{u, w\} + \beta\{v, w\}$

(9) $\{v, v\} = \sum_{k=1}^{N} \dfrac{v_k^{\,2} - \overline{v_k}^{\,2}}{2i} = \sum_{k=1}^{N} \mathrm{Im}\!\left(v_k^{\,2}\right) \in R .$

PROPOSITION 3　　　　The inner product $[v, w]$ satisfies

(10) $[v,w]=[w,v],$

(11) $[\alpha u+\beta v,w]=\alpha[u,w]+\beta[v,w],$

(12) $[v,v]\in C,$

where $u,v,w\in C^N$, and $\alpha,\beta\in C$.

PROOF.

(10) $[v,w]=\sum_{k=1}^{N}v_k w_k=\sum_{k=1}^{N}w_k v_k=[w,v].$

(11) $[\alpha u+\beta v,w]=\sum_{k=1}^{N}(\alpha u_k+\beta v_k)w_k=\alpha\sum_{k=1}^{N}u_k w_k+\beta\sum_{k=1}^{N}v_k w_k=\alpha[u,w]+\beta[v,w].$

(12) $[v,v]=\sum_{k=1}^{N}v_k{}^2\in C.$

The inner product $[v,w]$ is the sum of products of two complex numbers. It is an extremely simple

number and but it extends to a complex domain. Further, $[v,w]={}^t vw=\sum_{k=1}^{N}v_k w_k$ is regarded as a

complement to the ordinary inner product $(v,w)={}^t v\overline{w}=\sum_{k=1}^{N}v_k \overline{w_k}$ in a complex vector space.

Following the definitions of these inner products, we define the norms and distances in these new
spaces using these inner products. In general, the definitions of ordinary norms and distances are limited by
several well-known conditions. We, however, call a norm, distance, or metric hereafter in a more extended way,
because the concept of an ordinary norm or distance corrupt when one think a vector space that can violate the
triangle inequality.

DEFINITION 2

(13)   The norm $|v|$ of a vector $v\in C^N$ with $\langle v,v\rangle$ is $\sqrt{\langle v,v\rangle}$.

(14)   The norm $|v|$ of a vector $v\in C^N$ with $\{v,v\}$ is $\sqrt{\{v,v\}}$.

(15)   The norm $|v|$ of a vector $v\in C^N$ with $[v,v]$ is $\sqrt{[v,v]}$.

DEFINITION 3

(16)   The distance $d\langle v, w \rangle$ is $|v - w| = \sqrt{\langle v - w, v - w \rangle}$, where $v, w \in C^N$.

(17) The distance $d\{v, w\}$ is $|v - w| = \sqrt{\{v - w, v - w\}}$, where $v, w \in C^N$.

(18) The distance $d[v, w]$ is $|v - w| = \sqrt{[v - w, v - w]}$, where $v, w \in C^N$.

Each distance is symmetric for v and w.

PROPOSITION 4   The distances $d\langle v, w \rangle$, $d\{v, w\}$, and $d[v, w]$ satisfy

(13) $d\langle v, w \rangle = d\langle w, v \rangle$

(14) $d\{v, w\} = d\{w, v\}$

(15) $d[v, w] = d[w, v]$

PROOF.

(13) $d\langle v, w \rangle = \sqrt{\langle v - w, v - w \rangle} = \sqrt{\langle -(w - v), -(w - v) \rangle} = \sqrt{\langle w - v, w - v \rangle} = d(w, v)$.

(14) $d\{v, w\} = \sqrt{\{v - w, v - w\}} = \sqrt{\{-(w - v), -(w - v)\}} = \sqrt{\{w - v, w - v\}} = d\{w, v\}$.

(15) $d[v, w] = \sqrt{[v - w, v - w]} = \sqrt{[-(w - v), -(w - v)]} = \sqrt{[w - v, w - v]} = d[w, v]$.

Each distance is pseudo-metric in which multiple pairs of two vectors having the distance of zero can exist.

PROPOSITION 5   The distances $d\langle v, w \rangle$, $d\{v, w\}$, and $d[v, w]$ satisfy

(16) v and w that satisfy $v \neq w$ and $d\langle v, w \rangle = 0$ exist.

(17) v and w that satisfy $v \neq w$ and $d\{v, w\} = 0$ exist.

(18) v and w that satisfy $v \neq w$ and $d[v, w] = 0$ exist.

PROOF.

(16)   Let $v = (a_1 + ib_1, \cdots, a_N + ib_N)$ and $w = (c_1 + id_1, \cdots, c_N + id_N)$ that satisfy $v \neq w$ and

$$\sum_{k=1}^{N} \left\{(a_k - c_k)^2 - (b_k - d_k)^2\right\} = 0 \text{ then}$$

$$d\langle v, w\rangle = \sqrt{\langle v - w, v - w\rangle} = \sqrt{\sum_{k=1}^{N}\left\{\frac{(v_k - w_k)^2 + \overline{(v_k - w_k)^2}}{2}\right\}} = \sqrt{\sum_{k=1}^{N} \text{Re}\left\{(v_k - w_k)^2\right\}}$$

$$= \sqrt{\sum_{k=1}^{N} \text{Re}\left(v_k^2 - 2v_k w_k + w_k^2\right)} = \sqrt{\sum_{i=1}^{N}\left\{(a_k^2 - b_k^2) - 2(a_k c_k - b_k d_k) + (c_k^2 - d_k^2)\right\}}$$

$$= \sqrt{\sum_{k=1}^{N}\left\{(a_k - c_k)^2 - (b_k - d_k)^2\right\}} = 0.$$

(17)   Let $v = (a_1 + ib_1, \cdots, a_N + ib_N)$ and $w = (c_1 + id_1, \cdots, c_N + id_N)$ that satisfy $v \neq w$ and

$$\sum_{k=1}^{N} (a_k - c_k)(b_k - d_k) = 0 \text{ then}$$

$$d\{v, w\} = \sqrt{\{v - w, v - w\}} = \sqrt{\sum_{k=1}^{N}\left\{\frac{(v_k - w_k)^2 - \overline{(v_k - w_k)^2}}{2i}\right\}} = \sqrt{\sum_{k=1}^{N} \text{Im}\left\{(v_k - w_k)^2\right\}}$$

$$= \sqrt{\sum_{k=1}^{N} \text{Im}\left\{v_k^2 - 2v_k w_k + w_k^2\right\}} = \sqrt{\sum_{i=1}^{N}\left\{2a_k b_k - 2(a_k d_k - b_k c_k) + 2c_k d_k\right\}}$$

$$= \sqrt{2\sum_{k=1}^{N}(a_k - c_k)(b_k - d_k)} = 0.$$

(18)   Let $v = (a_1 + ib_1, \cdots, a_N + ib_N)$ and $w = (c_1 + id_1, \cdots, c_N + id_N)$ that satisfy $v \neq w$ and

$$\sum_{k=1}^{N}\left\{(a_k - c_k)^2 - (b_k - d_k)^2 + 2i(a_k - c_k)(b_k - d_k)\right\} = 0 \text{ then}$$

$$d[v, w] = \sqrt{[v - w, v - w]} = \sqrt{\langle v - w, v - w\rangle + i\{v - w, v - w\}}$$

$$= \sqrt{\sum_{k=1}^{N}\left\{(a_k - c_k)^2 - (b_k - d_k)^2 + 2i(a_k - c_k)(b_k - d_k)\right\}} = 0.$$

Moreover, each distance can be not only a real number, but also an imaginary number for $d\langle v, w\rangle$ and

$d\{v, w\}$, a complex number for $d[v, w]$.

PROPOSITION 6　　　　　The distances $d\langle v,w\rangle$, $d\{v,w\}$, and $d[v,w]$ satisfy

(19) If $\langle v-w,v-w\rangle \geq 0$, then $\mathrm{Re}(d\langle v,w\rangle)\neq 0$ and $\mathrm{Im}(d\langle v,w\rangle)=0$. If $\langle v-w,v-w\rangle < 0$, then

$\mathrm{Re}(d\langle v,w\rangle)=0$ and $\mathrm{Im}(d\langle v,w\rangle)\neq 0$.

(20)　　　If $\{v-w,v-w\}\geq 0$, then $\mathrm{Re}(d\{v,w\})\neq 0$ and $\mathrm{Im}(d\{v,w\})=0$. If $\{v-w,v-w\}<0$,

then $\mathrm{Re}(d\{v,w\})=0$ and $\mathrm{Im}(d\{v,w\})\neq 0$.

(21) $d[v,w]\in C$.

PROOF.

(19)　　　From (6) and $d\langle v,w\rangle = \sqrt{\langle v-w,v-w\rangle}$, if $\langle v-w,v-w\rangle \geq 0$, then $\mathrm{Re}(d\langle v,w\rangle)\neq 0$ and

$\mathrm{Im}(d\langle v,w\rangle)=0$. If $\langle v-w,v-w\rangle < 0$, then $\mathrm{Re}(d\langle v,w\rangle)=0$ and $\mathrm{Im}(d\langle v,w\rangle)\neq 0$.

(20)　　　From (9) and $d\{v,w\}=\sqrt{\{v-w,v-w\}}$, if $\{v-w,v-w\}\geq 0$, then $\mathrm{Re}(d\{v,w\})\neq 0$ and

$\mathrm{Im}(d\{v,w\})=0$. If $\{v-w,v-w\}<0$, then $\mathrm{Re}(d\{v,w\})=0$ and $\mathrm{Im}(d\{v,w\})\neq 0$.

(21)　　　From (12), $d[v,w]=\sqrt{[v-w,v-w]}\in C$.

Normally, distances should be zero or positive real numbers. These distances, however, are extended to negative real and imaginary domains.

　　　Here, we show several propositions related these metric spaces.

PROPOSITION 7　　　　　If $v,w\in R^N$, the metric space with $\langle v,w\rangle$ or $[v,w]$ is equal to an

N-dimensional Euclidean space.

PROOF.

If $v,w\in R^N$, $\langle v,w\rangle = \dfrac{{}^t vw + \overline{{}^t vw}}{2} = \dfrac{{}^t vw + {}^t vw}{2} = {}^t vw$ and $[v,w]={}^t vw$. Each of the metric spaces is

equivalent to a Euclidean space, which has a metric $(v,w)={}^t vw$.

PROPOSITION 8   If $v, w \in R^N$, the metric space with $\{v, w\}$ is identically the space that has a metric of 0.

PROOF.

If $v, w \in R^N$, $\{v, w\} = \dfrac{{}^t vw - \overline{{}^t vw}}{2i} = \dfrac{{}^t vw - {}^t vw}{2i} = 0$.

3. Isometric transformations

In metric spaces, the existence of their isometric transformations is critical. An orthogonal matrix T, a matrix that satisfies ${}^t TT = T {}^t T = E$, is only the isometric transformation for $\langle v, w \rangle$, $\{v, w\}$, and $[v, w]$ (as proved below). This is also true in Euclidean spaces, while unitary transformations are needed in complex vector spaces. Thus, the class of metrics in this study is regarded as a simple extension of the Euclidean metric from the point of view of isometric transformations, rather than the complex vector metric in which the unitary transformation is needed.

PROPOSITION 9   For arbitrary v and w,

(22) ${}^t TT = E \Longleftrightarrow \langle Tv, Tw \rangle = \langle v, w \rangle$

(23) ${}^t TT = E \Longleftrightarrow \{Tv, Tw\} = \{v, w\}$

(24) ${}^t TT = E \Longleftrightarrow [Tv, Tw] = [v, w]$

PROOF.

(22) ${}^t TT = E \Longrightarrow \langle Tv, Tw \rangle = \langle v, w \rangle$ for arbitrary v and w:

Let T to be an orthogonal transformation, then

$$\langle Tv, Tw \rangle = \frac{{}^t (Tv) Tw + \overline{{}^t (Tv)} \overline{Tw}}{2} = \frac{{}^t v {}^t TTw + \overline{{}^t v {}^t TT} \overline{w}}{2} = \frac{{}^t vw + \overline{{}^t vw}}{2} = \langle v, w \rangle.$$

For arbitrary v and w, $\langle Tv, Tw \rangle = \langle v, w \rangle \Longrightarrow {}^t TT = E$:

Suppose ${}^t TT \neq E$ and $\langle Tv, Tw \rangle = \langle v, w \rangle$ for arbitrary v and w. From

$$\langle Tv, Tw \rangle - \langle v, w \rangle = \langle Tv, Tw \rangle - \langle Ev, Ew \rangle = \langle (T+E)v, (T-E)w \rangle$$

$$= \frac{{}^t v^t (T+E)(T-E)w + \overline{{}^t v}\,{}^t (T+E)(T-E)\overline{w}}{2} = \frac{{}^t v C w + \overline{{}^t v}\, \overline{C}\, \overline{w}}{2}$$

$$= \sum_{j,k=1}^{N} \frac{c_{jk} v_j w_k + \overline{c_{jk}}\, \overline{v_j}\, \overline{w_k}}{2}, \quad \text{where } {}^t (T+E)(T-E) \text{ is replaced by } C, \text{ and then } \langle Tv, Tw \rangle = \langle v, w \rangle$$

becomes

$$(25) \qquad \sum_{j,k=1}^{N} \frac{c_{jk} v_j w_k + \overline{c_{jk}}\, \overline{v_j}\, \overline{w_k}}{2} = 0,$$

where c_{jk} ($\overline{c_{jk}}$) is an element of C ($\overline{C}$), and constant by choosing a T. But (25) states that the sum of

two quadratic forms of vectors $v, w, \overline{v}$ and $\overline{w}$ is identically 0, meaning that v and w are not arbitrary, and

contradicts the premise that states v and w are arbitrary. Therefore $\langle Tv, Tw \rangle = \langle v, w \rangle \wedge {}^t TT \neq E$ is false,

which leads $\langle Tv, Tw \rangle = \langle v, w \rangle \Rightarrow {}^t TT = E$ is true.

$(23) \qquad {}^t TT = E \Rightarrow \{Tv, Tw\} = \{v, w\}$ for arbitrary v and w:

Let T to be an orthogonal transformation, then

$$\{Tv, Tw\} = \frac{{}^t (Tv)Tw - \overline{{}^t (Tv)}\overline{Tw}}{2i} = \frac{{}^t v^t TTw - \overline{{}^t v}\, \overline{{}^t TT}\, \overline{w}}{2i} = \frac{{}^t v w - \overline{{}^t v}\, \overline{w}}{2i} = \{v, w\}.$$

For arbitrary v and w, $\{Tv, Tw\} = \{v, w\} \Rightarrow {}^t TT = E$:

Suppose ${}^t TT \neq E$ and $\{Tv, Tw\} = \{v, w\}$ for arbitrary v and w. From

$$\{Tv, Tw\} - \{v, w\} = \{Tv, Tw\} - \{Ev, Ew\} = \{(T+E)v, (T-E)w\}$$

$$= \frac{{}^t v^t (T+E)(T-E)w - \overline{{}^t v}\,{}^t (T+E)(T-E)\overline{w}}{2i} = \frac{{}^t v C w - \overline{{}^t v}\, \overline{C}\, \overline{w}}{2i}$$

$$= \sum_{j,k=1}^{N} \frac{c_{jk} v_j w_k - \overline{c_{jk}}\, \overline{v_j}\, \overline{w_k}}{2i}, \quad \text{where } {}^t (T+E)(T-E) \text{ is replaced by } C, \text{ and then } \{Tv, Tw\} = \{v, w\}$$

becomes

$$(26) \qquad \sum_{j,k=1}^{N} \frac{c_{jk} v_j w_k - \overline{c_{jk}}\, \overline{v_j}\, \overline{w_k}}{2i} = 0,$$

where c_{jk} ($\overline{c_{jk}}$) is an element of C ($\overline{C}$), and constant by choosing a T. But (26) states that the sum of two quadratic forms of vectors $v, w, \overline{v}$ and $\overline{w}$ is identically 0, meaning that v and w are not arbitrary, and contradicts the premise that states v and w are arbitrary. Therefore $\{Tv, Tw\} = \{v, w\} \wedge {}^t T T \neq E$ is false, which leads $\{Tv, Tw\} = \{v, w\} \Rightarrow {}^t T T = E$ is true.

(24) $\qquad {}^t T T = E \Rightarrow [Tv, Tw] = [v, w]$ for arbitrary v and w:

Let T to be an orthogonal transformation, then $[Tv, Tw] = {}^t(Tv)Tw = {}^t v {}^t T T w = {}^t vw = [v, w]$.

For arbitrary v and w, $[Tv, Tw] = [v, w] \Rightarrow {}^t T T = E$:

Suppose ${}^t T T \neq E$ and $[Tv, Tw] = [v, w]$ for arbitrary v and w. From

$$[Tv, Tw] - [v, w] = [Tv, Tw] - [Ev, Ew] = [(T + E)v, (T - E)w]$$
$$= {}^t v {}^t (T + E)(T - E)w = {}^t v C w$$

$$= \sum_{j,k=1}^{N} c_{jk} v_j w_k, \text{ where } {}^t(T + E)(T - E) \text{ is replaced by } C, \text{ and then } [Tv, Tw] = [v, w] \text{ becomes}$$

(27) $\qquad \sum_{j,k=1}^{N} c_{jk} v_j w_k = 0,$

where c_{jk} is an element of C, and constant by choosing a T. But (27) states that the quadratic form of vectors v and w is identically 0, meaning that v and w are not arbitrary, and contradicts the premise that states v and w are arbitrary. Therefore $[Tv, Tw] = [v, w] \wedge {}^t T T \neq E$ is false, which leads

$[Tv, Tw] = [v, w] \Rightarrow {}^t T T = E$ is true.

4. Violation of the triangle inequality

We finally conclude that the class of our metric spaces accepts both of the not-violated and violated triangle inequality. This property makes these metrics afford real applications. We start with the following lemma.

LEMMA 1.

(28) $\qquad \langle a, b \rangle = \frac{1}{2}\left(\langle a, a \rangle + \langle b, b \rangle - \langle c, c \rangle\right),$

(29) $\{a,b\} = \dfrac{1}{2}(\{a,a\}+\{b,b\}-\{c,c\})$,

(30) $[a,b] = \dfrac{1}{2}([a,a]+[b,b]-[c,c])$,

where $c = b - a$ and $a,b,c \in C^N$.

   PROOF.

(28) $\langle c,c \rangle = \langle b-a, b-a \rangle = \langle b,b \rangle + \langle -a,b \rangle + \langle b,-a \rangle + \langle -a,-a \rangle = \langle b,b \rangle - 2\langle a,b \rangle + \langle a,a \rangle$

Thus $\langle a,b \rangle = \dfrac{1}{2}(\langle a,a \rangle + \langle b,b \rangle - \langle c,c \rangle)$.

(29) $\{c,c\} = \{b-a, b-a\} = \{b,b\} + \{-a,b\} + \{b,-a\} + \{-a,-a\} = \{b,b\} - 2\{a,b\} + \{a,a\}$

Thus. $\{a,b\} = \dfrac{1}{2}(\{a,a\} + \{b,b\} - \{c,c\})$

(30) $[c,c] = [b-a, b-a] = [\langle b,b \rangle] + [-a,b] + [b,-a] + [-a,-a] = [b,b] - 2[a,b] + [a,a]$

Thus $[a,b] = \dfrac{1}{2}([a,a] + [b,b] - [c,c])$.

Using this lemma, we proof propositions in the following. The proofs are similar to those of a multi-dimensional scaling algorithm, which is often used in several scientific fields.

   PROPOSITION 10.

A set of different N points $\{a_i : i = 1, \cdots, N\}$, where the distance $d[a_i, a_j]$ between different two points

a_i, a_j are arbitrarily specified as complex numbers, can be generally positioned in a $(N-1)$-dimensional

complex vector space C^{N-1} with the metric $[a_i, a_j]$.

   PROOF.

Suppose $a_1 = (0, \cdots, 0)$ (we can arbitrarily set the relative location of the only one point of all) and Z is a

$(N-1)$-dimensional matrix that has an element $z_{ij} = [a_i, a_j] : (i, j = 2, \cdots, N)$ thus symmetric. From (30),

the   elements   of Z can   be   transformed   as

$z_{ij} = [a_i, a_j] = \dfrac{1}{2}([a_i, a_i] + [a_j, a_j] - [a_j - a_i, a_j - a_i]) = \dfrac{1}{2}(d_{1i}^2 + d_{1j}^2 - d_{ij}^2)$, where $d_{ij} = d[a_i, a_j]$.

In general, any matrix can be decomposed into an eigenvalue diagonal matrix and eigenvector matrices. After

the decomposition of Z, we obtain $Z = YL\,^tY$, where L is a diagonal matrix that has eigenvalues, Y is

an eigenvector matrix, and ^tY is its transposed matrix. Taking $\sqrt{L} = L^{\frac{1}{2}}$, we can get two matrix $^tX = YL^{\frac{1}{2}}$

and

(31) $\qquad X = L^{\frac{1}{2}\,t}Y$

from $Z = YL\,^tY = YL^{\frac{1}{2}}L^{\frac{1}{2}\,t}Y = {}^tXX$. This $(N-1)$ dimensional matrix X consists of the $(N-1)$

column vectors, which represents the coordinates of the $(N-1)$ points $a_i \in C^{N-1} : \{i = 2, \cdots N\}$ with the

origin $a_1 = (0, \cdots, 0) \in C^{N-1}$.

PROPOSITION 11.

A set of different N points $\{a_i : i = 1, \cdots, N\}$, where the distance $d\langle a_i, a_j \rangle$ between different two points

a_i, a_j are arbitrarily specified as real numbers in which the triangle inequality can either hold or not hold, can

be generally positioned in a $(N-1)$-dimensional complex vector space C^{N-1} with the metric $\langle a_i, a_j \rangle$.

PROOF.

Similar with the proof of PROPOSITION 10, suppose $a_1 = (0, \cdots, 0)$ and Z is a $(N-1)$-dimensional

matrix that has an element $z_{ij} = [a_i, a_j] : (i, j = 2, \cdots, N)$ thus symmetric. From (3),

$$Z = \begin{pmatrix} [a_2, a_2] & \cdots & [a_2, a_N] \\ \vdots & \ddots & \vdots \\ [a_N, a_2] & \cdots & [a_N, a_N] \end{pmatrix} = \begin{pmatrix} \langle a_2, a_2 \rangle & \cdots & \langle a_2, a_N \rangle \\ \vdots & \ddots & \vdots \\ \langle a_N, a_2 \rangle & \cdots & \langle a_N, a_N \rangle \end{pmatrix} + i \begin{pmatrix} \{a_2, a_2\} & \cdots & \{a_2, a_N\} \\ \vdots & \ddots & \vdots \\ \{a_N, a_2\} & \cdots & \{a_N, a_N\} \end{pmatrix} .$$

Therefore, if all the elements $z_{ij} = [a_i, a_j]$ are specified as real numbers, the resulted coordinate matrix X,

which is the same as (31), is of the metric $\langle a_i, a_j \rangle$. Apparently, if one can specify arbitrary real numbers as

the elements $z_{ij} = \frac{1}{2}(d_{1i}^2 + d_{1j}^2 - d_{ij}^2)$, points can either satisfy the triangle inequality or not.

PROPOSITION 12.

A set of different N points $\{a_i : i = 1, \cdots, N\}$, where the distance $d\{a_i, a_j\}$ between different two points

a_i, a_j are arbitrarily specified as real numbers in which the triangle inequality can either hold or not hold, can be generally positioned in a $(N-1)$-dimensional complex vector space C^{N-1} with the metric $\{a_i, a_j\}$.

PROOF.

Similar with the proof of PROPOSITION 10, suppose $a_1 = (0,\cdots,0)$ and Z is a $(N-1)$-dimensional matrix that has an element $z_{jk} = [a_j, a_k] : (j,k = 2,\cdots,N)$ thus symmetric. From (3),

$$Z = \begin{pmatrix} [a_2, a_2] & \cdots & [a_2, a_N] \\ \vdots & \ddots & \vdots \\ [a_N, a_2] & \cdots & [a_N, a_N] \end{pmatrix} = \begin{pmatrix} \langle a_2, a_2 \rangle & \cdots & \langle a_2, a_N \rangle \\ \vdots & \ddots & \vdots \\ \langle a_N, a_2 \rangle & \cdots & \langle a_N, a_N \rangle \end{pmatrix} + i \begin{pmatrix} \{a_2, a_2\} & \cdots & \{a_2, a_N\} \\ \vdots & \ddots & \vdots \\ \{a_N, a_2\} & \cdots & \{a_N, a_N\} \end{pmatrix} .$$

Therefore, if all the elements $z_{jk} = [a_j, a_k]$ are specified as imaginary numbers ($z_{jk} = i\{a_j, a_k\}$), the resulted coordinate matrix X, which is the same as (31), is of the metric $\{a_i, a_j\}$. Apparently, if one can specify arbitrary $i \times \text{real numbers}$ as the elements $z_{jk} = \frac{1}{2}\left(d_{1j}^2 + d_{1k}^2 - d_{jk}^2\right)$, points can either satisfy the triangle inequality or not, in the sense of $\text{Im}\left\{z_{jk} = \frac{1}{2}\left(d_{1j}^2 + d_{1k}^2 - d_{jk}^2\right)\right\}$.

5. Conclusion

We proposed a class of multi-dimensional feature spaces that can violate the triangle inequality by defining a class of metrics, norms, and distances. The class of our metric is pseudo-semimetric and it simultaneously affords vectors of both the mixed not-violated and violated triangle inequality. The violation or not are mathematically symmetric caused of the symmetry in real and imaginary numbers. Our space is exactly identical to Euclidean space if the imaginary parts of all the vectors are zero. We have demonstrated only orthogonal matrices are the class of isometric transformations in our metric spaces. We have further proved the propositions that vectors of which the distances between any pair are given can be always allocated to explicit positions with the corresponding coordinates in a multi-dimensional complex vector space by eigenvalue decomposition without any logical conflict. By only replacing real vectors to complex vectors, this theory is easily adopted to known algorithms in information science such as pattern recognition and neural networks.

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