Mathematical Analysis of Vapor Diffusion Process for Impregnated Solvent on Sheet-Type Films -Diffusion Status Model for Designing a Control System Configuration-

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Abstract

In this paper, we design a control system configuration in a drying oven using the mathematical model of the status in the drying oven and describe it using a transfer function with a quadratic time delay. The transfer function is called a vapor pressure function and has main state variables in a drying oven.

From a mathematical point of view, a one-dimensional advection diffusion equation (ODAE), for which the object model has a constant speed v in the direction, defines the drying oven status of the target system. The ODAE is given by applying the Lagrange differential operator. For example, it is possible to determine the optimal heat source position or the distribution function under ODAE.

It is generally difficult to measure the real data in the state variables within the drying oven. However, it is possible to determine the real data using various estimation algorithms such as Kalman filter theory and also to determine the configuration of a control system.

Keyword: thermal diffusion, Laplace transform, drying oven, impregnating solvent

1 Introduction

In general, a machine produces sheet type films. Its function is to first impregnate a solvent, especially an organic solvent, into the films, and then control the thickness of the impregnated film using a heat source in the drying oven. Such a machine is called "impregnating machine," and it produces shaped insulator films. Some companies have developed various types of machines that process many films using various solvents in a conventional manner. As a result, many different types of films have been produced.

In previous studies related to impregnating machines, it is widely recognized that the impregnated solvent on sheet-type films gets dried while diffusing the heat in a drying oven, or the sheet-type films may themselves move during the process of receiving heat in the form of steam in the drying oven[1, 2]. It is considered that the most important process units of the impregnating machine are the impregnating solvent and the drying unit.

Horiuchi et al. proposed a drying simulator which calculates the state variables that are

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related to the drying condition on impregnated films using the thermal diffusion in the drying oven[3]. In the case of this simulator, its function provides the data regarding these state variables using numerical analysis for one-dimensional partial differential equations (PDE) with unsteady states under both boundary conditions and process conditions. This excellent simulator is widely used in many studies as a drying mathematical model.

This paper proposes that the main state variables are given by an internal vapor pressure function to realize the design of a control system configuration. From the above description, the mathematical method uses a transfer function with a second-order time delay model as a more specific heat diffusion function in the drying oven. This paper also provides a mathematical model in which the impregnated films move with a velocity " v " in a certain direction. The ordinary differential equation is given by the Lagrange Differential Operator (LDO). This paper reports that the state of thermal diffusion to be discussed is defined by the ODAE[8].

First, this paper explains that the mathematical model is described using a transfer function with a second-order time delay model in which the object to be dried is stationary in the drying oven[7].

After obtaining the ODAE using Laplace transforms, the mathematical model of the thermal diffusion state equation is derived by LDO. Here, LDO is provided by the independent time variable t and dependent special variable x.

With respect to the direction of motion, assuming that the movement of films is in a onedimensional direction and that the impregnating solvent on the films is also dried in a one-dimensional direction by thermal diffusion, the model in a drying oven that describes the impregnated solvent on the films as a diffused continuous object in the direction of motion is presented by ODAE.

Instead of focusing on the internal reaction when impregnating films, we are to design a dynamic state model on a control system. To do this, it is necessary to derive the mathematical model that describes the situation in which the solvent vapor is diffused by the vapor pressure unit.

Further, with such a model, for example, it is also possible to determine the optimal heat source position or the distribution function. The measurement of the real state quantities is generally difficult. However, it allows the configuration of a control system when used to estimate the state variables using estimation tools[8].

As described above, the method proposed in this paper provides an easy mathematical model in ODEA.

2 Basic mathematical model and definition of the physical quantities

With respect to a model that remains stationary for both the films and heat source in the drying oven shown in Fig.1, the physical quantities used in this section are described as follows[3, 4, 6].
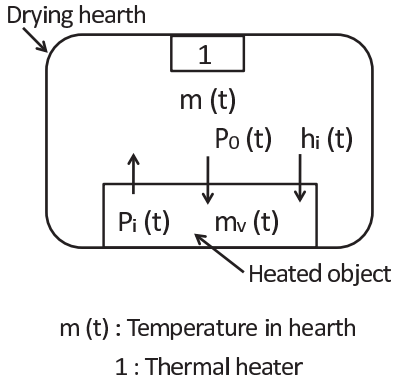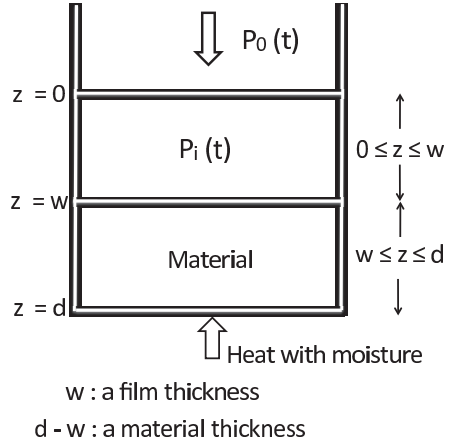
Fig.1: Static model in the drying oven

Fig.2: Oven standard model

Assuming that the vapor pressure is derived using the function of the moisture $q(t)$ in a drying oven, the vapor pressure is derived by

$$P_0(t) = g_1[q(t), \ t \geq 0] \tag{2.1}$$

where, the moisture function $q(t)$ is defined as the following equation:

Definition 1 *Moisture function $q(t)$*

$$
\begin{aligned}
q(t) &= \int r(t)dt \\
&= \int g_2[P_i(t) - P_0(t)]dt, \quad t \geq 0
\end{aligned} \tag{2.2}
$$

where $r(t)$ is the ratio of drying, $r(t) = g_2[P_i(t) - P_0(t)]$, $P_i(t)$ is the vapor pressure in the drying oven, and $P_0(t)$ is the external vapor pressure.

The vapor pressure derived by the gradient on the average moisture $m_v(t)$ on the films in the drying oven, is defined by

Definition 2 *Gradient on the average moisture $m_v(t)$ on the films*

$$P_i(t) = \frac{\partial P_i}{\partial m_v}\bigg|_{m_v = m_v^*} \tag{2.3}$$

where, $m_v(t)$ is derived by

$$m_v(t) = \int H[h_i(t) - h_0(t)]dt, \quad t \geq 0 \tag{2.4}$$

where $h_i(t)$ is the amount of heat transfer and $h_0(t)$ is the heat consumption. Now, $h_i(t)$ is described by

$$h_i(t) = [g_3(m_v(t)) - g_4(m_v(t))]dt, \quad t \geq 0 \tag{2.5}$$

where, H is the function representing a physical constraint.

$g_k(\bullet)$, $k = 1,2,3,4$ in Eqs.(2.1)-(2.5) represents the constraint function on the each physical quantity.

In this case, the transfer function $W(s)$ in such a drying oven is derived by

$$W(s) = \frac{Q(s)}{M(s)} \tag{2.6}$$

where $Q(s)$ is the Laplace transform of the moisture spring from the solvent obtained using the heat source in the dryin g oven, and $M(s)$ is the Laplace transform of the heat source function (the temperature in drying oven).

Such a thermal system model is generally derived as the time delay with a second-order system:

$$W(s) = K_G \frac{1}{a_3 + a_2 s + a_1 s^2} \tag{2.7}$$

where, let K_G, a_1, a_2, a_3 be a positive real number respectively.

From Eqs. (2.6)-(2.7), the following ODE is obtained formally by

$$a_1 \frac{d^2 q(t)}{dt^2} + a_2 \frac{dq(t)}{dt} + a_3 q(t) = K_G m(t) \tag{2.8}$$

where, the initial condition is $q(0) = q_0$[7].

The state variables on the object are considered as follows.

This is a mathematical model for which the films with the impregnated solvent represent the upper layer condition dried by the heat source under certain conditions (See Fig.2).

Since the moisture is approximately equal to the vapor pressure, we let the vapor pressure function be $C(x, t)$.

In addition, from Eq.(2.6), let the heat source (the temperature in the drying oven) again be $f(t)$. Then, $W(s)$ is derived by

$$W(s) = \frac{C(s)}{F(s)} \tag{2.9}$$

The model of Eq.(2.9) shows that the thickness of the impregnated solvent on films is proportional to the value of the moisture vapor pressure function based on time. In the case of a constraint condition such as heating the impregnating solvents, the thickness of the impregnated solvent on films decreases. However, the thickness of the films themselves maintains a constant value.

To mathematically model the continuous films, we assume that the films move with velocity

v in the direction.

To describe the mathematical model, the Lagrange differential operator D/Dt is introduced by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \tag{2.10}$$

where x denotes the spatial variable in the direction of movement[8].

From Eq.(2.8), Eq.(2.10) can be rewritten by

$$a_1\frac{D^2q(t)}{dt^2} + a_2\frac{Dq(t)}{dt} + a_3q(t) = K_Gm(t) \tag{2.11}$$

Similarly, Eq.(2.9) is derived by

$$\hat{a}_1\frac{D^2C(x,t)}{dt^2} + \hat{a}_2\frac{DC(x,t)}{dt} + \hat{a}_3q(t) = \hat{K}_Gf(t) \tag{2.12}$$

For the state variable below, it is assumed that Eq. (2.12), which is described by the internal vapor pressure function $C(x, t)$ is the target model system.

From Eqs.(2.10), (2.12) can be rewritten by

$$\hat{a}_1\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^2 C(x,t) + \hat{a}_2\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)C(x,t) + \hat{a}_3C(x,t) = Kf(x,t) \tag{2.13}$$

Assuming that the diffusion of moisture moves in one direction, by ignoring these terms $\left(\partial/\partial t\right)\cdot\left(\partial/\partial x\right)$ in Eq.(2.13), Eq.(2.14) is described as follows:

$$\begin{aligned}\frac{\partial C(x,t)}{\partial t} + v\frac{\partial C(x,t)}{\partial x} + C(x,t) &= Dv^2f(x,t)\\ &= D_c\frac{\partial^2 C(x,t)}{\partial x^2} + kf(x,t)\end{aligned} \tag{2.14}$$

Equation (2.14) represents ODAE, where Dc denotes the diffusion coefficient and $f(x, t)$ denotes the distribution function in the thermal diffusion state.

As described above, it can be expressed using the PDEs in Eq. (2.14), in which the movement model of the continuous films has a state variable of the internal vapor pressure.
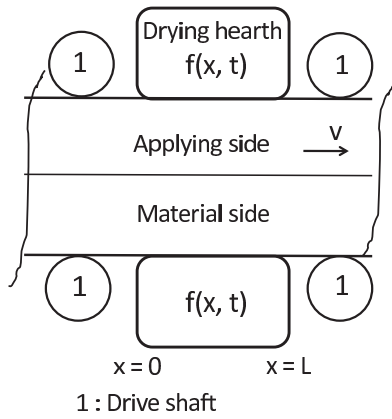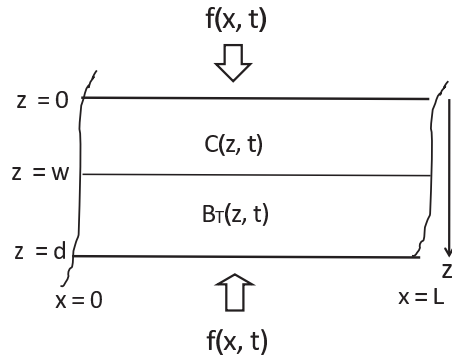
Fig.3: Plant model

Fig.4: Applied material model

3 Drying model of the sheet-type films to be moved

Figure 3 shows the model on this equipment.

Figure 4 shows the model on the impregnating solvent on the films.

Let $C(z, t)$ be the internal vapor pressure function in the direction of the z-axis (dry direction), and $B_T(z, t)$ denotes the transfer function on the film side. Let $z \in [w, d]$, $t \in [0, T]$.

Figure 4 indicates the movement model for continuous films with the impregnated solvent at a moving velocity v while being dried by diffusion vapor. We assume that the thickness of impregnated solvent T_s decreases with the ratio of $\partial T_s / \partial t$ on account of the solvent vapor.

The ratio of $\partial T_s / \partial t$ is derived by

$$\frac{\partial T_s}{\partial t} = -\nu_1 \cdot r, \quad t \in [0, T] \tag{3.1}$$

where ν_1 denotes the inverse parameter of the solvent density, and r is the drying ratio parameter proposed by Clausius? Clapeyron Equation in chemical thermodynamics[4, 5, 6].

Equation (3.1) represents the constraint equation for the change in thickness of the impregnated solvent on films.

Here, the impregnated solvent thickness of the films is defined by

Definition 3 *Impregnating solvent thickness on films*：$T_{s_0} \leq z \leq w$, where T_{s_0} is

$$T_{s_0} = \left. \frac{\partial T_s}{\partial t} \right|_{t=0}. \tag{3.2}$$

Definition 4 *Film thickness*：$w \leq T_m \leq d$, where T_{m_0} is

$$T_{m_0} = \left. \frac{\partial T_m}{\partial t} \right|_{t=0}. \tag{3.3}$$

With respect to the movement direction, let $x = x_n$ at any point x, and with respect to the z-axis direction, the state variable $C(z, x, t)$ is derived by

$$\frac{\partial C(z,x_n,t)}{\partial t} = D(z)\frac{\partial^2 C(z,x_n,t)}{\partial z^2} \tag{3.4}$$

$$C(z,x_n,t) \equiv C(z,t), \quad {}^{\forall}x = x_n \tag{3.5}$$

With respect to the z-axis direction, Eq.(3.4) is

$$\frac{\partial C(z,t)}{\partial t} = D(z)\frac{\partial^2 C(z,t)}{\partial z^2} \tag{3.6}$$

$$C(z,0) \equiv C(z), \quad T_{s_0} \leq z \leq w \tag{3.7}$$

$$\lambda_1 \frac{dC(z)}{dz} = -r\Delta H + h_a[f(t) - C(z)] \tag{3.8}$$
$$\cong r_c C(z)$$

where the boundary condition is $z = 0$, $-r \cdot \Delta H$ is the latent heat of the solvent vapor, h_a is the thermal conductivity in the impregnated solvent direction, and r_c is a general parameter.

The impregnated solvent thickness is represented by

$$\frac{\partial T_s}{\partial t} = -\nu_1 \cdot r, \quad z = 0 \tag{3.9}$$

$$\frac{\partial C(z,t)}{\partial z} = 0, \quad z = w \tag{3.10}$$

where the boundary equations for the continuous condition at the boundary surface $z = w$ were obtained by

$$\lambda_1 \frac{\partial C(z,t)}{\partial z}\bigg|_{z=w} = \lambda_2 \frac{\partial B_T(z,t)}{\partial z}\bigg|_{z=w} \tag{3.11}$$

$$C(w,t) = B_T(w,t) \tag{3.12}$$

where both λ_1 and λ_2 are a thermal conductivity respectively.

With respect to the film side, both the dynamic model and initial condition of B_T (z, t) are obtained by

$$\frac{\partial B_T(z,t)}{\partial t} = \alpha_B \frac{\partial^2 B_T(z,t)}{\partial z^2} \tag{3.13}$$

$$B_T(z,0) = B_T(z) \tag{3.14}$$

where α_B denotes a diffusion coefficient.

According to the set at $z = d$ in Eq. (3.8), it is similarly obtained by

$$\lambda_2 \frac{\partial B_T(z,t)}{\partial z} = h_b[f(t) - B_T(z,t)]$$
$$\cong r_B \cdot B_T(z,t) \tag{3.15}$$

where h_b is the thermal conductivity to the outside in a drying oven, and r_B is called the general parameter.

Assuming that the change in the thickness of the impregnated solvent is generally constrained by

Assumption 1

$$\frac{\partial T_s}{\partial t} = \eta(\nu, r), \quad t \geq 0 \tag{3.16}$$

where, $\eta(\bullet)$ is assumed to be uniformly decreasing function. The diffusion equation that describes the impregnated solvent on films in the z-axis direction is generally given by

$$\frac{\partial C(z,t)}{\partial t} = D(z)\frac{\partial^2 C(z,t)}{\partial z^2} + \delta(x - x_m)f(t) \tag{3.17}$$

where $\delta(x - x_m)$ denotes the Dirac Delta function distribution and $x = x_m$ denotes the distribution point of the heat source.

With respect to $\delta(x - x_m)$, $\delta(x - x_m) = \varphi(x, x_m)$, $\varphi(x, x_m)$ can be represented by the distribution function of C^∞[8]. According to Eq. (3.16), the impregnated solvent thickness change is represented by

$$\frac{\partial T_s}{\partial \theta} = \eta[C(z,\theta), \nu(\theta), r], \quad \theta \geq 0 \tag{3.18}$$

where $\theta = x/\nu$ and $\theta = 0$, then $z^* = 0$.

With respect to the impregnated solvent thickness, z is described by

$$T_s = \int_0^\theta \eta[C(z,\theta), \nu(\theta), r]d\theta, \quad \theta \geq 0 \tag{3.19}$$

where z denotes a decreasing function in Fig.5, and can be approximated by a straight line $z = \sqrt{x^2 + w^{*2}}$. From this linearization, the change in the direction of x can be regarded as being equivalent to the change in the direction of θ. Assuming that the term $C(x, t)$ in Eq. (2.14)
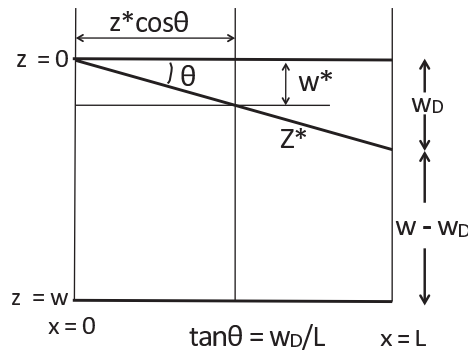

Fig.5: Approximate linear model for applying film thickness with changes

is omitted, the new mathematical model Eq. (2.14) is rewritten as

$$\frac{\partial C(z,\theta)}{\partial \theta} + v\frac{\partial C(z,\theta)}{\partial z} = D(z)\frac{\partial^2 C(z,\theta)}{\partial z^2} + B_1(z)\cdot f(\theta) \tag{3.20}$$

where, $z \in [z^*, w]$ and $\theta \in [0, T]$.

The mathematical model of film side is rewritten by

$$\frac{\partial B_T(z,\theta)}{\partial \theta} + v\frac{\partial B_T(z,\theta)}{\partial z} = D(z)\frac{\partial^2 B_T(z,\theta)}{\partial z^2} + B_2(z)\cdot f(\theta) \tag{3.21}$$

where, $z \in [w, d]$ and $\theta \in [0, T]$.

The boundary equation at the borderline $z = w$ between the solvent and film is obtained by

$$\lambda_1\frac{\partial C(z,\theta)}{\partial z}\bigg|_{z=w} = \lambda_2\frac{\partial B_T(z,\theta)}{\partial z}\bigg|_{z=w} \tag{3.22}$$

where, both $B_1(z)$ and $B_2(z)$ denote the distribution function for the control function $f(\theta)$ respectively.

This paper proposes a general diffusion model as follows:

$$\frac{\partial C(z,\theta)}{\partial \theta} + v\frac{\partial C(z,\theta)}{\partial z} = D(z)\frac{\partial^2 C(z,\theta)}{\partial z^2} + B(z)\cdot f(\theta) \tag{3.23}$$

where $z \in [z^*, L]$, $\theta \in [0, T]$.

T_s can be derived by

$$T_s = \int_0^\theta \eta[C(z,\theta),v(\theta),r]d\theta, \quad \theta \geq 0 \tag{3.24}$$

where $C(0,\theta) = C_0(\theta)$ and $C(w, \theta) = C_w(\theta)$.

In case of $z = 0$, the boundary condition is derived by

$$\frac{\partial C(z,\theta)}{\partial z}\bigg|_{z=0} \cong \beta_1 \cdot C_0(\theta), \quad ^\forall\beta_1 > 0 \tag{3.25}$$

In case of $z = w$, the boundary condition is derived by

$$\frac{\partial C(z,\theta)}{\partial z}\bigg|_{z=w} \cong \beta_2 \cdot C_w(\theta), \quad ^\forall\beta_2 > 0 \tag{3.26}$$

where, assuming that $\theta \approx 0$ because of the extremely thin nature of the impregnated solvent, it can be regarded as x in the same way as z^*. From Fig.5, let $x = z^* \cos\theta$.

In the case where there is focus on the diffusion model along the direction of motion, the most generalized mathematical equation is derived by

$$\frac{\partial C(x,t)}{\partial t} + v\frac{\partial C(x,t)}{\partial x} = D(x)\frac{\partial^2 C(x,t)}{\partial x^2} + \varphi(x,x_m)\cdot f(t) \tag{3.27}$$

where, $C(x, 0) = C_0(x)$, $x \in [0, L]$, $t \in [0, T]$.

The boundary conditions at both $x = 0$ and $x = L$ in Eq.(3.27) are respectively obtained by

$$\frac{\partial C(x,t)}{\partial x}\Big|_{x=0} = \beta_1\xi_1(t), \quad \beta_1 > 0 \tag{3.28}$$

$$\frac{\partial C(x,t)}{\partial x}\Big|_{x=L} = \beta_2\xi_2(t), \quad \beta_2 > 0 \tag{3.29}$$

where both ξ_1 and ξ_2 denote the constraint functions on each boundary position at $x=0$ and $x=L$, respectively.

That is, with respect to Eq.(3.27)～Eq.(3.29), the vapor pressure function $C(x, t)$ to the impregnated solvent on the film represents the one-dimensional diffusion status along the direction of motion. When the vapor pressure function denotes $C(L, T)$ at $x = L$, $t = T$, the impregnated solvent becomes w^*. If a constraint function on the solvent thickness is specified, the study revealed that the impregnated solvent thickness can be controlled by $C(x, t)$.

In this paper, the characteristics of the mathematical model revealed the following with respect to both the stationary model and the diffusion model, respectively:

- It is converted from the transfer function model to the diffusion model by the Lagrange differential operator.

- Assuming that the impregnated solvent thickness decreases uniformly under the extremely thin layer of solvent while impregnating to films, the diffusion of the x-axis direction in the same thickness equates as the diffusion of z axis direction.

4 Examples

【Example-1】 Stationary state model

The mixed boundary condition problem, which is a combination of the Neumann and Dirichlet conditions.

$$\frac{\partial C(x,t)}{\partial t} = \alpha\frac{\partial^2 C(x,t)}{\partial x^2}$$

where, let $C_0(x) = 1.0$, $x \in [0, 1]$, $t \in [0, 1]$.

For this example, the boundary condition, the eigenvalue and the eigenfunction are respectively given by

$$C(0,t) = C(1,t) = 0$$

$$\lambda_i = -\alpha\left(\frac{2i-1}{2}\right)\pi$$

$$\varphi_i(x) = \sqrt{2}\sin\left(\frac{2i-1}{2}\right)\pi x$$

$$x_{0i} = \int_0^1 C_0(x)\varphi_i(x)dx$$
$$= \left(\frac{2\sqrt{2}}{(2i-1)\pi}\right)\left\{1 - \cos\left(\frac{2i-1}{2}\right)\pi\right\}$$

【Example-2】　Stationary state model

Neumann type boundary condition problem.

$$\frac{\partial C(x,t)}{\partial t} = \alpha\frac{\partial^2 C(x,t)}{\partial x^2}$$
$$\frac{\partial C(x,t)}{\partial x}\bigg|_{x=0,1} = 0$$

where let $C(x, t) = C_0(x)$.

$$\lambda_i = -\alpha(i\pi)^2$$
$$\varphi_i(x) = \sqrt{2}\cos(i\pi)x$$

Exactly,

$$\lambda_i = \alpha\left\{(m-1)\pi\right\}^2$$
$$\varphi_m(x) = \sqrt{2}\,cos\left\{(m-1)\pi\right\}x, \quad m \geq 2$$

【Example-3】　ODAE model

The definition related to the diffusion coefficient is

Definition 5　*Definition of diffusion coefficient D*

$$D \equiv \frac{\kappa}{\rho C} \tag{4.1}$$

where, κ denotes a transfer coefficient of films, C, and $\rho(x)$ the transport rate. Eq.(4.1) is rewritten as follows[9]:

$$\frac{\partial C(x,t)}{\partial t} + \rho\frac{\partial C(x,t)}{\partial x} = D\frac{\partial^2 C(x,t)}{\partial x^2} \tag{4.2}$$

The analytical particular solutions of ODAE is derived by

$$C(x,t) = \frac{L}{\rho}\frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right) \tag{4.3}$$

where, $C(x, t)$ denotes the diffusion vapor function, and let $x \in [0, L]$ and $t \in [0, T]$.

With respect to Eq.(4.3), the initial condition and the boundary condition are respectively

$$C(x,0) \equiv C^0(x)\delta(x), \tag{4.4}$$
$$\delta(x) = \begin{cases} \delta(x) = 1 & (x = 0) \\ \delta(x) = 0 & (except\ above) \end{cases} \tag{4.5}$$

$$\frac{\partial C(x,t)}{\partial x}\Big|_{x=L} = aC^L(t) \tag{4.6}$$

$$\frac{\partial C(x,t)}{\partial x}\Big|_{x=0} = bC^0(t) \tag{4.7}$$

where, a and b represent constants. $C^L(t)$ is the state amount at the end of the diffusion vapor, and $C^0(x)$ is the state amount at the beginning of the diffusion vapor.

As described above, the equation to constrain the diffusion vapor function $C(x, t)$ is to be written in ODAE.

To perform a normalized nondimensional equation, the following variables in Eq. (4.2) are translated as follows:

$$\tau = \frac{1}{L}\rho t, \ \ l = \frac{x}{L}, \ \ \mu = \frac{\rho L}{D} \tag{4.8}$$

From Eq.(4.8), normalized non-dimensional equation is obtained by

$$\frac{\partial C(l,\tau)}{\partial \tau} + \frac{\partial C(l,\tau)}{\partial l} = \frac{1}{\mu}\frac{\partial^2 C(l,\tau)}{\partial l^2} \tag{4.9}$$

where, assuming that ρ is a constant velocity, let $\tau \in [0,1], \ l \in [0,1]$.

To translate the model in Eq. (4.9) into the mode domain, $C(l, \tau)$, which is translated by the eigenfunction expansion, is obtained by

$$C(l,\tau) = \sum_{j=1}^{\infty} U_i^j(\tau)\varphi_i^j(l) \tag{4.10}$$

See the Appendix A, B for further detail[?].

$$U_i^j(t) = \int_0^1 \zeta(l)C(x,t)\varphi_i^j(x)dx \tag{4.11}$$

where, $\zeta(l)$ は, where, $\zeta(l)$ is

$$\zeta(l) = \exp(-\mu l). \tag{4.12}$$

According to Eq.(4.12), in order to perform a self-adjoint operation on Eq.(4.9), the model and boundary conditions for the related eigenvalues are

$$\frac{1}{\mu}\frac{d}{dl}\Big\{\zeta(l)\frac{d\varphi_i(l)}{dl}\Big\} + \lambda_i\zeta(l)\varphi_i(l) = 0 \tag{4.13}$$

$$\Big\{a\varphi_i(l) + \frac{d\varphi_i^j(l)}{dl}\Big\}_{l=1} = 0 \tag{4.14}$$

$$\Big\{b\varphi_i(l) + \frac{d\varphi_i^j(l)}{dl}\Big\}_{l=0} = 0 \tag{4.15}$$

From Eq.(4.13)～Eq.(4.15), both the eigenfunction $\varphi_i^j(l)$ and eigenvalue λ_i^j can be obtained by

$$\varphi_i^j(l) = A_i^j \exp\Big(\frac{\mu l}{2}\Big)\Big\{\Big(\frac{2\alpha_i^j}{\mu}\Big)\cos\alpha_i^j l + \sin\alpha_i^j l\Big\} \tag{4.16}$$

$$\lambda_i^j = \frac{\mu}{4} + \frac{\alpha_i^{j2}}{\mu} \tag{4.17}$$

where A_i^j denotes the constant value that defines the set of orthogonal function systems in the eigenfunction $\varphi_i^j(l)$. See the Appendix A,B for further detail.

$$A_i^j = \left\{ 2 \left(\frac{\alpha^2_i}{\mu^2} + \frac{1}{\mu} + \frac{1}{4} \right) \right\}^{-\frac{1}{2}} \tag{4.18}$$

where, α_i denotes the solution of the transcendental equation.

$$\cot \alpha_i^j = \frac{\alpha_i^{j2}}{\mu} - \frac{\mu}{4\alpha_i^j} \tag{4.19}$$

With respect to the parameters in the tables, Table.1，2，3 represent the eigenvalue and the eigenfunction-related diffusion parameters. As the eigenvalues become larger, the diffusion coefficient also increases. The setting parameters in Fig.6 use the data shown in Table.1. The setting parameters in Fig.7 use the data shown in Table.2. The setting parameters in Fig.8 use the data shown in Table.3. (1) in Fig.6～Fig.8 show that the vapor diffusion density U_i^j $(l,\ \tau)$ is expanded by the eigenfunctions with up to three eigenvalues. (2) in Fig.6～Fig.8 represents the vapor diffusion value $\frac{\partial U_i^j(l,\tau)}{\partial l}$. (3) in Fig.6～Fig.8 represents the eigenvalues in Eq.(4.17).

Table. 1: Set parameter values

$\mu = 0.5$		
$\alpha_i^1 = 0.69$	$A_i^1 = 0.346921$	$\lambda_i^1 = 1.08$
$\alpha_i^2 = 3.29$	$A_i^2 = 0.104776$	$\lambda_i^2 = 21.77$
$\alpha_i^3 = 6.36$	$A_i^3 = 0.055208$	$\lambda_i^3 = 81.02$

Table. 2: Set parameter values

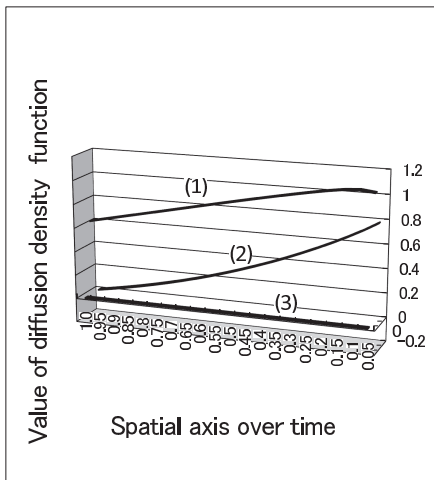| $\mu = 1.0$		
$\alpha_i^1 = 0.96$	$A_i^1 = 0.479838$	$\lambda_i^1 = 1.17$
$\alpha_i^2 = 3.43$	$A_i^2 = 0.196004$	$\lambda_i^2 = 12.01$
$\alpha_i^3 = 6.45$	$A_i^3 = 0.108018$	$\lambda_i^3 = 41.85$

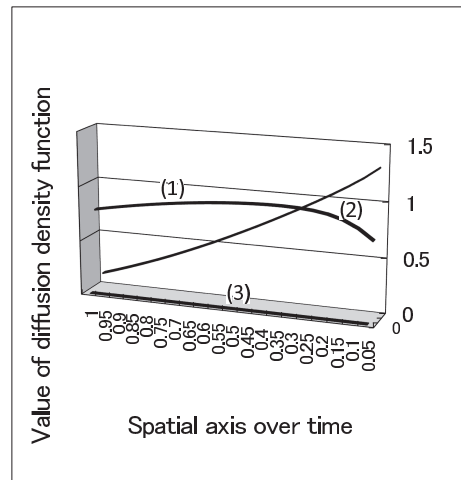*Fig.6:* Schematic diagram of diffusion vapor function

Fig.7: Schematic diagram of diffusion vapor function

Table. 3: Set parameter values

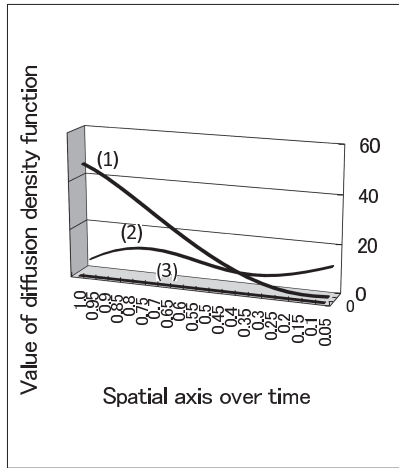| $\mu = 0.5$		
$\alpha_i^1 = 1.78$	$A_i^1 = 0.931100$	$\lambda_i^1 = 1.88$
$\alpha_i^2 = 4.17$	$A_i^2 = 0.660658$	$\lambda_i^2 = 4.73$
$\alpha_i^3 = 6.95$	$A_i^3 = 0.458147$	$\lambda_i^3 = 10.91$

Fig.8: Schematic diagram of diffusion vapor function

5 Results

With respect to the model proposed in this paper, Examples 1 and 2, involving state models that are stationary, are derived by the one-dimensional diffusion equation, and Example 3 is ODAE. All examples could be obtained analytically for a solution which denotes an internal thermal diffusion situation under no external force.

In the case of the drying simulator proposed by Horiuchi et al., it is widely recognized that the values measured by the prototype machine agree well with simulation results.

According to the transfer function obtained by the analytical method in this paper, a mathematical model which can be described using PDEs can be obtained easily to represent the internal model in a drying oven.

For applications in a realistic scenario, it is necessary to identify both the various parameters and functions. However, on the basis of the control theory results derived by the various PDE models, the method proposed by this paper is shown to be useful.

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A Appendix1：Derivation of eigenfunction

According to substitute $\zeta(l) = e^{-\mu l}$ into Eq.(4.13) and run the differential calculus, it can be obtained by

$$e^{-\mu l}\left\{\frac{d^2\varphi_i^j(l)}{dl^2} - \mu\frac{d\varphi_i^j(l)}{dl} + \mu\lambda_i\varphi_i^j(l)\right\} = 0, \tag{A.1}$$

In order to establish Eq.(A.1), the following equation must be satisfied:

$$\frac{d^2\varphi_i^j(l)}{dl^2} - \mu\frac{d\varphi_i^j(l)}{dl} + \mu\lambda_i\varphi_i^j(l) = 0 \tag{A.2}$$

According to get a particular solution by Laplace transform, it can be obtained a particular solution from Eq.(A.2) as follows.

$$\mathscr{L}\left\{\frac{d^2\varphi_i^j(l)}{dl^2}\right\} = s^2\Phi_i(s) - s\varphi_i^j(0) - \varphi_i^{j'}(0) \tag{A.3}$$

$$\mathscr{L}\left\{\frac{d\varphi_i^j(l)}{dl}\right\} = s\Phi_i(s) - \varphi_i^j(0) \tag{A.4}$$

According to substitute Eqs. (A.3), (A.2) into Eq.(A.2), it can be obtained by

$$\Phi_i(s) = \frac{s}{s^2-\mu s+\mu\lambda_i}\varphi_i^j(0) + \frac{1}{s^2-\mu s+\mu\lambda_i}\{\mu\varphi_i^j(0)+\varphi_i^{j'}(0)\} \tag{A.5}$$

From Eq. (??A.5)，

$$\mu\varphi_i^j(0) + \varphi_i^{j'}(0) = 0 \tag{A.6}$$

The solution in Laplace transformation is obtained by

$$\Phi_i(s) = \frac{s}{s^2 - \mu s + \mu \lambda_i} \varphi_i^j(0)$$ (A.7)

Let use Inverse Laplace transform Eq.(A.8)

$$
\begin{aligned}
&\mathscr{L}^{-1}\left\{\frac{s}{s^2 + 2bs + c}\right\} \\
&= \frac{s+b}{(s+b)^2 + (\sqrt{c-b^2})^2} - \frac{b}{(s+b)^2 + (\sqrt{c-b^2})^2} \\
&= e^{-bt}(\cos rt - \frac{b}{r}\sin rt), (r = \sqrt{c-b^2})
\end{aligned}
$$ (A.8)

According to Eq. (A.8), let b $= -\mu/2$, c $= \mu \lambda_i$

$$
\begin{aligned}
\varphi_i^j(l) &= \varphi_i^j(0)e^{\frac{\mu}{2}l}\{\cos\alpha_i l + (\frac{\mu}{2\alpha_i})\sin\alpha_i l\}, \\
&= \varphi_i^j(0) \cdot \frac{\mu}{2\alpha_i} \cdot e^{\frac{\mu}{2}l}\left\{\left(\frac{2\alpha_i}{\mu}\right)\cos\alpha_i l + \sin\alpha_i l\right\}
\end{aligned}
$$ (A.9)

After setting $\varphi_i^j(0) \cdot \frac{\mu}{2\alpha_i} = A_i$ in Eq.(A.9), the eigenfunction is obtained

B Appendix2: Induction of coefficient with respect to eigenfunction

According to substitute the eigenfunction $\psi_i^j(l)$ into Eq.(B.1), it is obtained by

$$\frac{1}{\mu} \cdot \frac{d}{dl}\left\{\zeta(l)\frac{d\varphi_i^j(l)}{dl}\right\} + \lambda_i\zeta(l)\varphi_i^j(l) = 0.$$ (B.1)

The second order differential equation (B.1) can be solved. According that the eigenfunction in this paper form the orthogonal function, it is obtained by

$$\int_0^1 \zeta(l) \cdot \varphi_i^j(l) \cdot \varphi_i^j(l)dl = 1$$ (B.2)

where, let $\zeta(l) = e^{-\mu l}$. After substituing Eq.(B.2) into $\psi_i^j(l), \zeta(l)$ and calculating Eq.(B.2), it is obtained by

$$
\begin{aligned}
&A_i^2\left\{\frac{2\alpha_i^2}{\mu^2} + \frac{2}{\mu} + \frac{1}{2} + (\frac{2\alpha_i^2}{\mu^2} - \frac{1}{2\alpha_i})\sin\alpha_i\cos\alpha_i - \frac{2}{\mu}\cos^2\alpha_i\right\} \\
&= 1
\end{aligned}
$$ (B.3)

In order to become one of the left side in Eq.(B.3), both Eq.(eq:f43) and Eq.(B.5) must be satisfied.

$$(\frac{2\alpha_i^2}{\mu^2} - \frac{1}{2\alpha_i})\sin\alpha_i\cos\alpha_i - \frac{2}{\mu}\cos^2\alpha_i = 0$$ (B.4)

Eq.(B.5) must be established.

$$A_i^2 \left\{ \frac{2\alpha_i^2}{\mu^2} + \frac{2}{\mu} + \frac{1}{2} \right\} = 1 \tag{B.5}$$

where, A_i and α_i have to satisfy Eqs. (B.6) and (B.7) respectively.

$$A_i = \left\{ 2\left(\frac{\alpha_i^2}{\mu^2} + \frac{1}{\mu} + \frac{1}{4} \right) \right\}^{-\frac{1}{2}} \tag{B.6}$$

$$\cot \alpha_i = \frac{\alpha_i^2}{\mu} - \frac{\mu}{4\alpha_i} \tag{B.7}$$