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SELF-SIMILARITY OF FLUCTUATIONS FOR THROUGHPUT DEVIATIONS WITHIN A PRODUCTION PROCESS

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Abstract. *We clarify the self-similarity of fluctuations in a supply chain system and present a size-independent mathematical model of a supply chain system using Langevin-type stochastic differential equations. As numerical examples, we provide density spectra (criteria of synchronization processes) for a given frequency ranges (lead time). The self-similarity of fluctuations is given by the function of throughput deviations within the process. We also demonstrate that for this supply chain system, when the time constant of the time correlation function possesses a uniform Poisson distribution, the system exhibits f^{-1} fluctuation and when this time constant possesses a uniform distribution, the system exhibits f^{-2} fluctuation. Furthermore, the supply chain system has a Lorentzian spectrum under the condition of fluctuations having spectral density. We also verify the self-similarity in the supply chain system. The probability distribution of cost rate based on the lead time exhibits a normal distribution. Furthermore, the probability distribution for the absolute value of the cost rate deviation based on the lead time exhibits a power-law distribution. Finally, regarding the management strategy to be taken by the manufacturer, we propose that profit can be increased when adopting a strategy that purposefully leads to a state of excessive production or one of excessive order entries.*
Keywords: Self-similarity, Fluctuation, Power-law distribution, Throughput deviation, Spectral density

1. Introduction.

Conventionally, in physics, it is known that, assuming that the scaling law holds as is the case with a phase transition phenomenon, the probability density function becomes power-law distribution [1]. Due to occurrence of an unforeseen situation in an economic phenomenon such as a stock price fluctuation and a yen-dollar exchange fluctuation, fast and furious volatility of stock prices or rapid fluctuation of yen-dollar exchange is caused. Further, also in the information communication network field, it has been reported that, as a result of similar data analysis about first and furious traffic fluctuations, it becomes power-law distribution [2, 3, 4, 5]. It is widely known that a field of econophysics as we know it today has been established.

With respect to a production process in manufacturing industry, as a result of analysis based on data of a rate of return of companies and its deviation collected over 10 years or more, we have noticed that such data has random variation [7]. By performing the data analysis, relation between a value of rate-of-return deviation and a production throughput became clear to some extent. "Fluctuation model of rate-of-return deviation" is of self-similarity, and it shows fractal nature [5, 6, 12]. Also, this power-law distribution characteristic has "fluctuation" nature in phase transition [5, 6]. For example, occurrence of fluctuation and the like were made clear through recognition as a phase transition point.

In a previous study [14], we addressed the problem of reducing construction work and inventory in the steel industry. Specifically, we investigated the relationship between variations in the rate of construction and delivery rate. In this study, the authors perform analysis using the queuing model and apply log-normal distribution to model the system in the steel industry [14].

Moreover, several studies have reported approaches that lead to shorter lead times [15, 16]. From order products, lead time occurs on the work required preparation of the members for manufacturing.

Many aspects can potentially affect lead time. For example, from order products, the lead time from the start of development to the completion of a product is called the time-to-finish time, such as the work required preparation of the members for production equipments. Moreover, several studies have focused on reducing customer lead times. In [17], the author addresses the problem of reducing the production lead time.

In [18], the authors propose a method that increases both production efficiency and production of a greater diversity of products for customer use. Their proposed approach results in shortened lead times and reduces the uncertainty in demand. Their method captures the stochastic demand of customers and produces solutions by solving a nonlinear stochastic programming problem.

In summary, several studies have considered uncertainty and proposed practical approaches to shorten the lead time. The demand is treated as a stochastic variable and apply mathematical programming. To our knowledge, previous studies have not treated lead time as a stochastic variable. Because fluctuations in the supply chain and market demand and the changes in the production volume of suppliers are propagated to other suppliers, their effects are amplified. Therefore, because the amounts of stock are large, an increase or decrease of the suppliers' stock is modeled using differential equation.

In our previous research, we propose the following conclusion. If an amount of money of order entries and an amount of money of production are stochastic, accumulated excessive order entries becomes of Brownian motion, and thus a random "fluctuation" occurs in hour to hour order entries and production even though it might be of a small degree. In comparison with a case where production is made to conform to the average order entry, profit can be increased in a case where strategy to purposefully lead to excessive production or excessive order entries state is adopted [7].

On the other hand, in a previous study of a production process, we constructed a state in which the production density of each process corresponds to the physical propagation of heat [19]. Using this approach, we showed that a diffusion equation dominates the production process [8]. Moreover, we made it clear that manufacture of products proceeds in multiple stages from the beginning of production. Such volatility is encountered in every stage of manufacturing, and the delays in the production line propagate this volatility to the successive step. A delay in the production process is equivalent to a "fluctuation" in physical phenomena [11]. To reduce lead-times of a production system, we propose using a mathematical model that focuses on the selection process and adaptation mechanism of the production lead time [9]. We model the throughput time of the production demand/manufacturing system in the manufacturing stage by using a stochastic differential equation of log-normal type, which is derived from its dynamic behavior. Using this model and the risk-neutral integral, we define and compute the evaluation equation for the compatibility condition of the production lead time. Furthermore, we apply the synchronization process and show that the throughput of the manufacturing process is reduced [10, 12].

With respect to production method, we described the differences between the synchronous and asynchronous models and showed that the throughput of a manufacturing process depends on volatility [13]. Synchronization implies that the machines and assembly lines manufacture the required production volumes in accordance with timing requirements. Moreover, to understand the difference between the asynchronous method, which causes a delay in the production process, and the synchronous method, which reduces the process throughput time in production processes, we manufactured equipment, That is, the synchronous method is the best way to product equipments [13].

In this study, we clarify the self-similarity of production processes by calculating spectral densities across a range of frequencies. We suggest that the self-similarity of fluctuations is proportional to throughput deviations within a process. Here frequency denotes lead time – an essential factor to determine the lead-time. To maintain the lead time in production processes, the processes must be properly synchronized. We suggest that a criterion of synchronization processes is suitable for spectral density and verify the self-similarity of fluctuations through numerical examples.

We report on the existence of the self-similarity of these fluctuations and note the f^{-1} and f^{-2} fluctuations. We also verify self-similarity in the system through experiments on the supply chain system. We have been producing control equipment using the supply chain system. Nine workers in total are involved, and the production process is composed of six stages. To compare the form of production, we roughly carry out four patterns of asynchronous and synchronous methods.

In the analysis results of the cost rate data based on the lead time, the probability distribution, which represents cost rate versus the deviation of cost rate based on the lead time, exhibits a normal distribution. Furthermore, the probability distribution for the absolute value of the cost rate deviation based on the lead time exhibits a power-law distribution. The power-law distribution suggests the existence of self-similarity in the production process.

In this report, regarding the management strategy to be taken for a manufacturer, we propose that it is possible to increase profit by adopting a strategy that purposefully leads to a state of excessive production or one of excessive order entries. This management strategy is ideal on the basis of analysis of the cost rate of the production process. To the best of our knowledge, the self-similarity of fluctuations in a supply chain system has not been previously reported.

2. **Production Systems in the Manufacturing Equipment Industry.** The production methods used in manufacturing equipment are briefly covered in this paper. More information is provided in our report. This system is considered to be a "Make-to-order system with version control", which enables manufacturing after orders are received from clients, resulting in "volatility" according to its delivery date and lead time. In addition, there is volatility in the lead time, depending on the content of the make-to-order products (production equipment).

In Figure 1(A), the "Customer side" refers to an ordering company and "Supplier (D)" means the target company in this paper. The product manufacturer, which is the source of the ordered manufacturing equipment presents an order that takes into account the market price. In Figure 1(B), the market development department at the customer's factory receives the order through the sale contract based on the predetermined strategy.

3. **Supply Chain System.** We study a process structure to increase throughput by dynamically changing the production system.

Figure 2 illustrates a company's decision-making process. The business monitors perceived demand trends. When a customer order is received, the perceived trend is analyzed. Based on the analysis, the company is able to decide how to respond to the analyzed demand.

Therefore, we require a dynamic supply chain management model. A diagram focusing on the supply chain between an assembly manufacturer and a parts suppliers is shown in

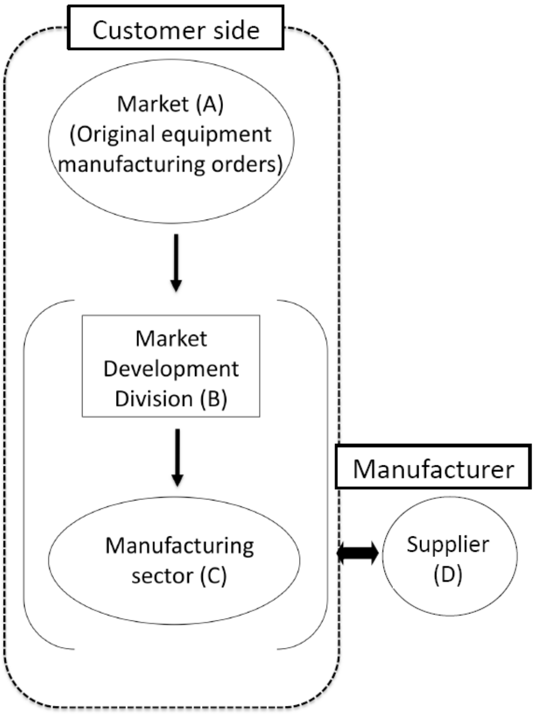

FIGURE 1. Business structure of company of research target

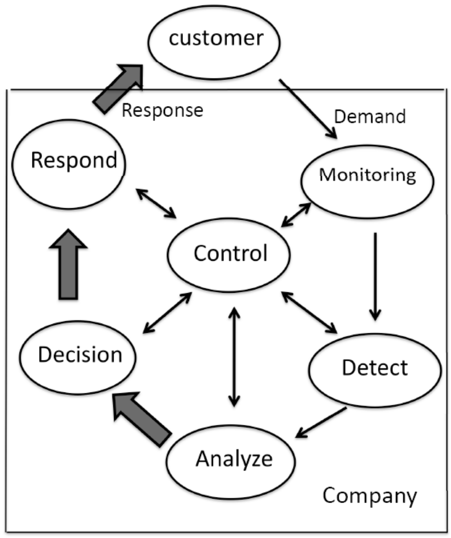

FIGURE 2. Decision-making process within the company

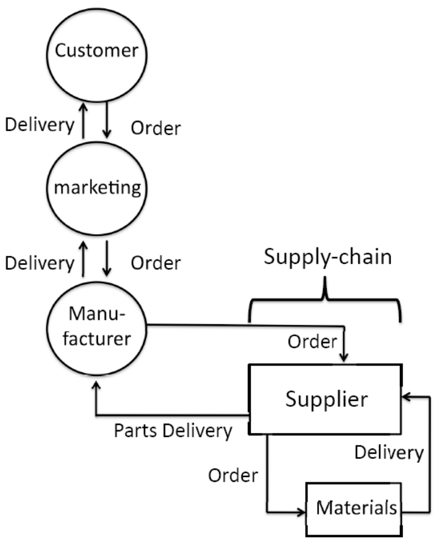

FIGURE 3. Supply chain management area

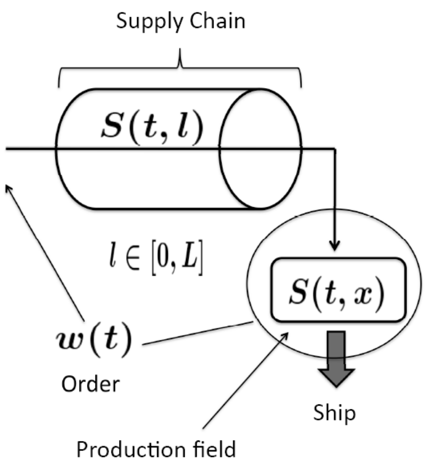

FIGURE 4. Supply chain and production field

Figure 3. We propose a complex supply chain model and a stochastic field (referred to as production field) in the production process (see Figure 4).

We propose that the production processes from the completion of product to delivery to the customer are similar to a continuous time model of thermal diffusion in physics. From this, we obtain the following.

To produce a product, the target company designs a product, orders materials from a parts supplier, initiates production, and then ships the product to a customer. A series of such operations leads to an inherent time delay. We stochastically analyze under the assumption that it is not necessarily a flow of deterministic information. This flow of a series of operations is simply a supply chain [10].

4. **Distribution Characteristics of Throughput Deviations within a Process.** We derive the diffusion structure model as follows:

$$\frac{\partial h(t,x)}{\partial t} = \mathcal{L}_{t,x} h(t,x) + b(t,x)h(t,x)Z(t,x) \tag{1}$$

$$\begin{aligned}\partial h(t,x) &= \mathcal{L}_{t,x} h(t,x)\partial t + b(t,x)h(t,x)Z(t,x)\partial t \\ &= \mathcal{L}_{t,x} h(t,x)\partial t + b(t,x)h(t,x)\partial W(t,x)\end{aligned} \tag{2}$$

where $h(t,x) \equiv C(t+1,x) - C(t,x)$.

Equation (2) derives the diffusion model of throughput deviations within a process.

$$\frac{\partial h(t,x)}{\partial t} = [\alpha + \xi(t,x)]h(t,x) + D^2 \frac{\partial^2 h(t,x)}{\partial x^2} \tag{3}$$

$$< \xi(t,x) \cdot \xi(t^{'},x^{'}) >= 2\delta(t-t^{'})\delta(x-x^{'}) \tag{4}$$

Equation (3) is rewritten as a stochastic partial differential equation (PDE) with advection as follows:

$$\frac{\partial h(t,x)}{\partial t} + v\frac{\partial h(t,x)}{\partial x} = D^2 \frac{\partial^2 h(t,x)}{\partial x^2} + b(t,x)h(t,x)Z(t,x) \tag{5}$$

By rewriting Equation (5), we are able to obtain the following:

$$\partial h(t,x) = \left[D^2 \frac{\partial^2 h(t,x)}{\partial x^2} - v\frac{\partial h(t,x)}{\partial x}\right]\partial t + b(t,x)h(t,x)Z(t,x) \tag{6}$$

In other words, "fluctuation" is caused by noise, which depends on the state of process throughput deviation. From Equation (3), we treat as a random variable rather than a growth rate (trend factor). In this study, the target mathematical model is treated as having added noise, which depends on state variables (state function) that are inherent in the process.

From the above description, we present a stochastic PDE for our model (Equation (6)). We also describe a phase transition for the same target system. The mathematical model that describes phase transition conforms to a study proposed by Yamamoto et al. [21].

Here we describe the Ginzburg-Landau free energy (G-L free energy) in a manufacturing industry as follows.

Definition 4.1. *Free energy: $F(S_i)$ related to production quantity*

$$F(h) = \int_0^L \left[\frac{r}{2}(\nabla h)^2 + W(h)\right] dx \tag{7}$$

As previously explained, order parameter $S_i(x,t)$ is a variable that is related to phase transition. Equation (7) indicates that free energy given by the space integration of a function depends on order parameter $S_i(x,t)$ and is G-L free energy. In addition, $f(S_i)$ indicates a potential function, and ∇S_i represents fluctuations.

Here a state where Equation (7) becomes minimum in processes is found. When $S_i(x,t)$ is changed by $\delta S_i(x,t)$, free-energy changes $\delta F(S_i)$ are as follows, where the space variable $x = [0, L]$ is one-dimensional.

$$W(h) = \frac{1}{8}h^4 - \frac{1}{4}h^3, \quad h > 0 \tag{8}$$

Then, in closed system, we describe the following:

$$\frac{dF(h)}{dt} = \int_0^L \frac{\delta F(h)}{\delta h} \cdot \frac{\partial h}{\partial t} dx \leq 0 \tag{9}$$

where $\delta F(h)/\delta h$ indicates Furesshe differential of $F(h)$.

$$\frac{\partial h}{\partial t} = -L(h)\frac{\delta F(h)}{\delta h}, \quad L(h) > 0 \tag{10}$$

If Equation (10) is satisfied, Equation (9) is able to satisfy [22], where a boundary condition holds until the completion of the entire process. Within the entire process cycle, throughput is as follows, because Equation (11) is satisfied within the entire process.

$$\frac{\partial h}{\partial \nu} = 0, \quad \nu \in [0, T] \times [0, L] \tag{11}$$

From Equation (11), we obtain the following:

$$\frac{\partial h}{\partial t} = rL(h) \cdot \nabla h + \frac{1}{2}L(h)h(1 - h^2) \tag{12}$$

where the model with the consideration of noise is as follows:

$$\partial h = rL(h) \cdot \nabla h \partial t + \frac{1}{2}L(h)h(1 - h^2)\partial t + b(t,x)h\partial W(t,x) \tag{13}$$

Then, if $L(h) \equiv 1$, we obtain the following:

$$\partial h(t,x) = r \cdot \nabla h(t,x)\partial t + \frac{1}{2}h(t,x)(1 - h^2(t,x))\partial t + b(t,x)h(t,x)\partial W(t,x) \tag{14}$$

From above description, the model that indicates a phase transition can be derived as Equation (14).

5. **Self-Similarity Model of Fluctuations.** We refer to Equation (3) as a throughput model. The throughput deviations are regarded as a function of time only in Equation (15).

$$\frac{dh(t)}{dt} + r(t)h(t) = f(t) \tag{15}$$

where $f(t)$ represents the nonlinearity of the production processes and noise due to nonlinearity with other suppliers.

Thus, Equation (15) becomes a well-known Langevin-type stochastic differential equation [23].

In this study, we show that a throughput rate is a model specified by a Langevin-type stochastic differential equation regardless of the size of the supply chain system. In other words, we indicate the self-similarity of the supply chain system.

Such a model can be explained as follows. An inter-arrival time in the supply chain system introduces fluctuation. The inter-arrival time is a Poisson process with time uniformity when the fluctuation is small. However, fluctuation affects the entire process when complex arrival processes become large. We illustrate the supply chain concept in Figure 5.

As shown in Figure 5, we define scale transform as follows.

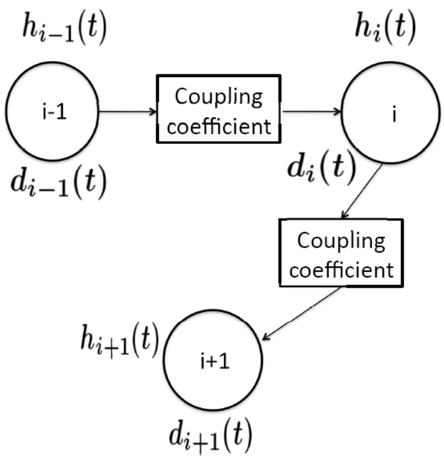

FIGURE 5. Supply chain system concept diagram

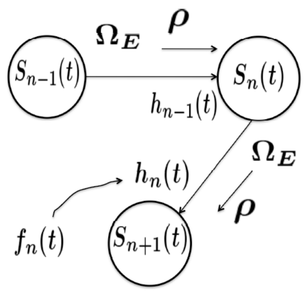

FIGURE 6. Supply chain throughput model

Definition 5.1. *Scale transform*

1. $a \equiv \Omega$: *System scale of a supply chain system (constant).*
2. $b \equiv \rho/\Omega_E$: *(2) Movement speed in accordance with the system scale of the supply chain (constant).*
3. $t' = \rho/\Omega_E \cdot t$: *Time.*
4. ρ: *(4) Rate of import of materials depending on the scale of the supply.*

This supply chain model is shown in Figure 6.

Then, according to the model shown in Figure 6, we assume the following relationship between S_{n-1} and S_n.

Assumption 1. *Following equation*

$$\sqrt{\frac{\Omega \cdot \Omega_E}{\rho}} h_{n-1}\left(\frac{\rho}{\Omega_E} \cdot t\right) = h_n(t) \tag{16}$$

Assumption 2. *A factor related to noise*

$$\sqrt{\rho \frac{\Omega}{\Omega_E}} f_{n-1}\left(\frac{\Omega}{\Omega_E} \cdot t\right) = f_n(t) \tag{17}$$

From Equation (16), we obtain as follows:

$$\sqrt{\frac{\Omega \cdot \Omega_E}{\rho}} h_0\left(\frac{\rho}{\Omega_E} \cdot t\right) = h_1(t) \tag{18}$$

Therefore, we obtain the following:

$$\sqrt{\frac{\Omega \cdot \Omega_E}{\rho}} \frac{dh_0\left(\frac{\rho}{\Omega_E} \cdot t\right)}{dt} = \frac{dh_1(t)}{dt} \tag{19}$$

Rewriting Equation (19), we obtain as follows (Refer Appendix A):

$$\frac{dh_0\left(\frac{\rho}{\Omega_E} \cdot t\right)}{dt} = \frac{\Omega_E}{\rho} \cdot \sqrt{\frac{\rho}{\Omega_E \Omega}} \frac{dh_1(t)}{dt} = \sqrt{\frac{\Omega_E}{\rho\Omega}} \frac{dh_1(t)}{dt} \tag{20}$$

From Equation (20), we obtain the following:

$$\frac{dh_1(t)}{dt} + r\left(\frac{\rho}{\Omega_E}\right)h_1(t) = f_1(t) \tag{21}$$

Then we obtain as follows:

$$\left[\begin{array}{ll} \frac{dh_0(t)}{dt} + rh_0(t) & = f_0(t) \\ \frac{dh_1(t)}{dt} + \left(\frac{\rho}{\Omega_E}\right) rh_1(t) & = f_1(t) \\ \vdots & \\ \frac{dh_n(t)}{dt} + \left(\frac{\rho}{\Omega_E}\right)^n rh_n(t) & = f_n(t) \end{array}\right. \tag{22}$$

where $n = 1, 2, \cdots$.

From the above scaling method, we can indicate the self-similarity of process throughput deviation [20]. We can then obtain the following after calculating $h_n(t)$.

$$\begin{aligned} h_n(t) = {} & \exp(-d^n rt)\Big[\Omega \cdot d\Big]^{n/2} \cdot h_n(0) \\ & + \Big[\Omega \cdot d\Big]^{n/2} \int_0^t dt' \exp(-d^n r(t-t')) \cdot f_0^n(t'), \quad {}^\forall d \equiv \rho/\Omega_E, \quad n = 1, 2, \cdots \end{aligned} \tag{23}$$

Then, we set the throughput deviation of a cumulating process as $S(h_n)$. Therefore, $S(h_n)$ indicates a power-law distribution [7, 20].

Definition 5.2. *Power-law distribution of $S(h_n)$.*

$$S[h_n] \cong r[\Omega \cdot d]^{n/2} h_n^{-\alpha[\rho/\Omega_E]^n}, \quad r > 0, \quad \alpha > 0 \tag{24}$$

Definition 5.3. *Following equation*

$$\left[\begin{array}{c} W_0^n \equiv d^n \cdot r \\ T_n(\cdot) \equiv [\Omega \cdot d]^{n/2} \cdot h_n(0) \\ a_n \equiv [\Omega \cdot d]^{n/2} \end{array}\right. \tag{25}$$

We rewrite Equation (23) as follows:

$$h_n(t) = \exp(-W_0^n(t)) \cdot T_n(\cdot) + \int_0^t dt' \exp(-W_0(t-t')) \cdot a_n f_n(t') \tag{26}$$

Accordingly, the mean square is expressed by the following.

$$\begin{aligned} < |h_n(t)|^2 >= {} & \exp(-2W_0^n \cdot t) < |T_n(0)|^2 > + < \left|\int_0^t dt' \exp(-W_0^n(t-t')a_n^{-1} f_n(t'))\right|^2 > \\ & + 2\exp(-2W_0^n \cdot t)\int_0^t dt' \exp(-W_0^n(t-t'))a_n^{-1} < T_n(0) \cdot f_n(t') > \end{aligned} \tag{27}$$

Equation (27) indicates that the initial value is $T_n(0)$, which has no correlation with $f_n(t)$ at time later than $t' > 0$.

Therefore, we obtain as follows:

$$< T_n(0) \cdot f_n(t') >=< T_n(0) > \cdot < f_n(t') >$$

Thus, $< f_n(t') = 0 >$, then we obtain the following:

$$\begin{aligned} < |h_n(t)|^2 >= {} & \exp(-2W_0^n \cdot t) < |T_n(0)|^2 > \\ & + < \left|\int_0^t dt' \exp(-W_0^n(t-t')a_n^{-1} f_n(t'))\right|^2 > \end{aligned} \tag{28}$$

Here random noise characteristics are indicated as follows.

$$\left[\begin{array}{c} < f_n(0) >= 0 \\ < f_n^{\eta}(t) \cdot f_n^{\eta'}(t') >= 2D_f \delta_{\eta\eta'}(t-t') \end{array}\right. \tag{29}$$

where the followings are satisfied.

$$\delta_{\eta\eta'} = \left\{\begin{array}{lcl} 1 & : & \eta = \eta' \\ 0 & : & \eta \neq \eta' \end{array}\right. \tag{30}$$

where D_f represents the intensity of random forces.

We can provide the following for a fluctuation under the equilibrium state [25].

$$< |h_n(t)|^2 >= \exp(-2W_0^n \cdot t) < |T_n(0)|^2 > +a_n^{-1} D_f \{1 - \exp(-2a_n^{-1}) \cdot t\} \tag{31}$$

6. **Spectrum Analysis of Throughput Deviations within a Process.** We focus on a cycle period as follows: In Figure 7, $T_{si}(t)$ indicates a cycle of period i, and $T_\rho(t)$ represents a company's fiscal year. Then, the relationship between $T_{si}(t)$ and $T_\rho(t)$ is expressed as follows.

$$T_{si}(t) = \frac{1}{h_n(t)} \tag{32}$$

When $T_\rho(t) \equiv f_{req}$, we refer to f_{req} as the company's period frequency.

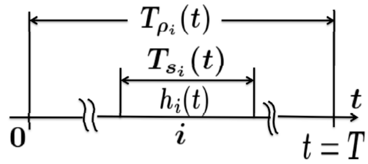

FIGURE 7. Conceptual model of process cycle period and duration

Then, from the Wiener-Khinchin theorem, a power spectrum $S_{h_n}(f_{req})$ of $h_n(t)$ to f_{req} is expressed by the following [24].

$$S_{h_n}(f_{req}) \cong \int_0^\infty \cos(2\pi f_\rho \cdot t)\phi_{h_n}(t)dt \tag{33}$$

$$\phi_\tau(t) \cong \int_0^\infty \cos(2\pi f_{req} \cdot t) S_{h_n}(f_{req}) df_{req} \tag{34}$$

Then we obtain as follows:

$$S_{h_n}(\tau) \approx D_f < |h_n|^2 > \tag{35}$$

Here if a time constant exists in the time correlation function of fluctuation, we can derive the following [24].

$$\phi_{h_n}(t) = D_f < |h_n|^2 > \cdot \exp\left(-\frac{t}{\tau_n}\right) \tag{36}$$

Then, we substitute Equation (36) into Equation (33) for $S_{h_n}(f_{req})$. Thus, we can obtain the following after calculating Equation (33) (see Appendix B).

$$S_{h_n}(f_{req}) =< |h_n|^2 > \cdot \frac{D_f \tau_n}{(2\pi f_{req} \tau_n)^2 + 1} \tag{37}$$

With respect to Equation (37), Equation (35), which provides a spectral correlation function of throughput, has a Lorentzian spectrum in the vicinity of the frequency of the fluctuation (see Appendix B).

If the time constant τ_n is proportional to τ_n^{-1}, we can obtain the following. Here, if the time constant τ_n is proportional to τ_n^{-1}, we can obtain the following:

$$\begin{aligned} < S_{h_n}(f_{req}) > &= \int_0^\infty \frac{d\tau_n}{\tau_n} \cdot \frac{D_{f_req}\tau_n}{(2\pi f_{req}\tau_n)^2+1} < |h_n|^2 > \\ &= D_{f_{req}} \int_0^\infty \frac{< |h_n|^2 >}{(2\pi\xi_n)^2+1} \cdot \frac{d\xi_n}{f_{req}}, \quad {}^\forall\xi_n = f_{req}\cdot\tau \end{aligned} \tag{38}$$

Here we calculate the average of Equation (38) with respect to the time constant τ_n and the distribution of the power spectrum of $h_n(t)$. Then, Equation (38) indicates f^{-1} fluctuation.

From f^{-1} fluctuations in Equation (38), because the input process or the correlation function spectrum of throughput has a Lorentzian spectrum in the vicinity of the frequency of fluctuations, and because the time constant distribution of the time correlation function has fluctuations between the manufacturer and the material supplier, the time constant distribution is proportional to τ_n^{-1}.

Then, if the distribution of τ_n is uniform, e.g., it equals the constant r, we can obtain the following.

$$\begin{aligned} < S_{h_n}(f_{req}) > &= \int_0^\infty \frac{D_{f_{req}}d\tau_n}{(2\pi f_{req}\tau_n)^2+1} \cdot r_n d\tau_n < |h_n|^2 > \\ &= D_{f_{req}} \int_0^\infty \frac{r_n\tau_n}{(2\pi f_{req}\tau_n)^2+1} d\tau_n \cdot < |h_n|^2 > \\ &= D_{f_{req}} \int_0^\infty \frac{r_n\xi_n}{(2\pi f_{req}\xi_n)^2+1} \cdot d\tau_n \cdot \frac{1}{f_{req}^2} d\xi \cdot < |h_n|^2 > \end{aligned} \tag{39}$$

Equation (39) represents f^{-2} fluctuations.

Considering the above description, we consider a time constant distribution in the following cases.
(1) The distribution of time constant τ_n is proportional to τ^{-1}.
(2) The distribution of time constant τ_n is uniform distribution.
Here (1) indicates f^{-1} fluctuations, and (2) indicates f^{-2} fluctuations. Thus, if a throughput model can be described by Equations (22) $-$ (26), we can better explain the characteristics of f^{-1} or f^{-2} fluctuations. f represents the number of process cycles during a period.

7. **Numerical Simulation.** Frequency represents lead time; it is important to determine frequency. For example, Figure 8, Figure 9 and Figure 10 show numerical examples of power spectral density versus frequency and represent the magnitude of throughput deviations within a process at a certain range of frequencies. The similarity of the graph shapes, which show power spectral densities at a certain frequency ranges, also indicates self-similarity in the supply chain.

By setting a certain range of frequencies, i.e., setting a target lead time, spectral density is maintained as low as possible. In other words, throughput deviations within a process are maintained as low as possible.

By maintaining high-throughput deviations within a process in the supply chain system, a company produces high benefits.

In Figure 8, the random intensity is 1, time constant is 0.5, and process spectral values denote frequencies.

In Figure 9 and Figure 10, a random intensity is = 1, a time constant is 1, and process spectral values show for frequencies.

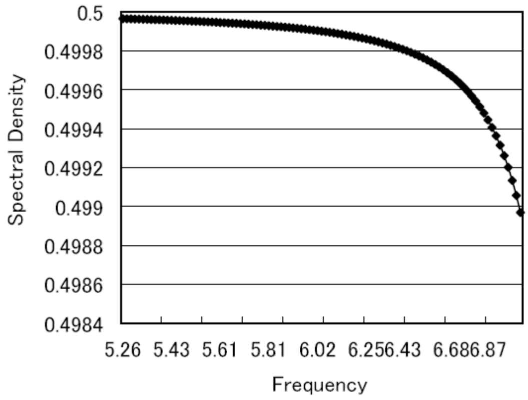

FIGURE 8. Power spectrum in the production process (frequency)

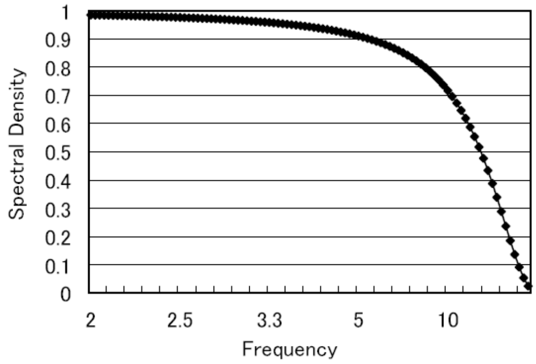

FIGURE 9. Power spectrum in the production process (frequency)

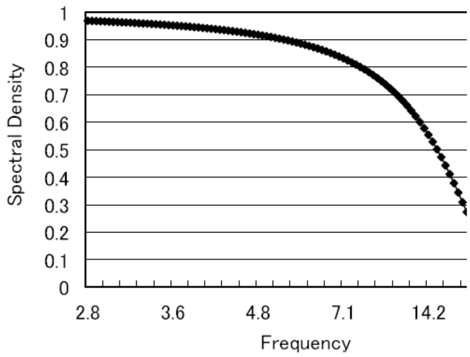

FIGURE 10. Power spectrum in the production process (frequency)

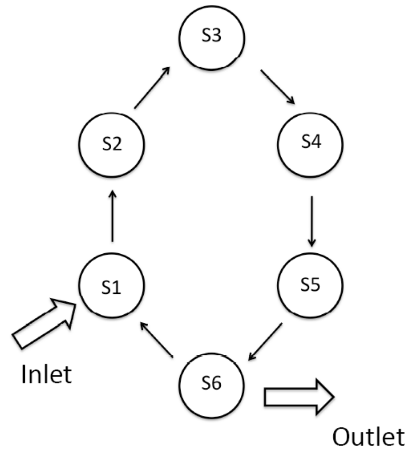

FIGURE 11. Flow production process

8. **Flow Production Process.** Figure 11 shows an example of a flow production process – a process commonly employed in the production of control equipment. In this example, the flow production process consists of six stages. In each step of the production process (i.e., S1 $-$ S6), materials are being produced as a supply chain system.

The direction of the arrows represents the direction of the production flow. In this process, production materials are supplied through the inlet, and the end-product is shipped from the outlet. We studied four production patterns, each of which consisted of an asynchronous process (constructed by non-skilled workers) and a synchronous process (constructed by skilled workers).

9. **Numerical Example.** We introduce an example of self-similarity.

Figure 12 and Figure 13 are the graphs which use the table data divided by 200 based on the lead time. Therefore, we have $54 \times 4 = 216$ data. We calculate the probability for all cost rate deviations where in each table based on the lead time, WC represents the labor cost that acts as a guideline. The unit of every cost data is yen/min (Table 2-Table 5). In Table 2-Table 5, each table has 54 data ((K1, S1)-(K9, S6)).

Figure 12 represents the normal distribution for the deviation of cost rate based on the lead time, where the average is zero and the standard deviation is 0.118. Figure 13 represents the power-law distribution (solid line), which indicates the existence probability

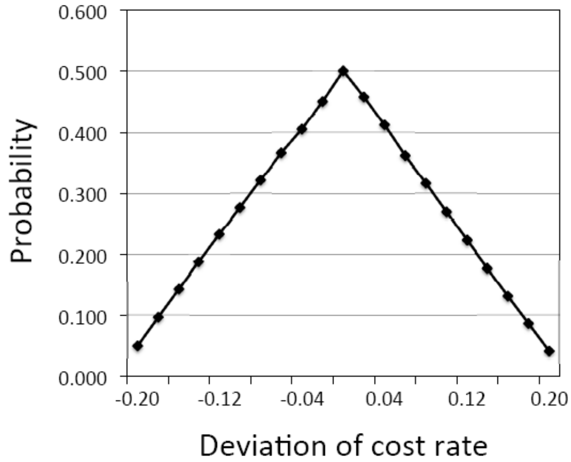

FIGURE 12. Normal distribution of the deviation of cost rate

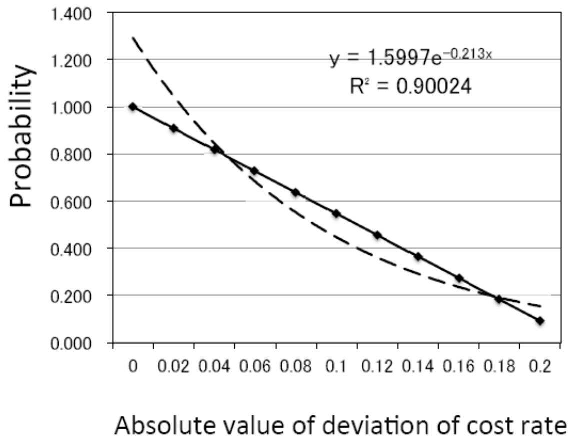

FIGURE 13. Power-law distribution and approximate curve (doted line)

TABLE 1. Descriptive statistics of the cost data

Mean	0.305
SD	0.036
Variance	0.019
Skewness	−0.805
Kurtosis	−0.482
SEM	0.45
Number of Data	15

on the vertical axis, shows the absolute value of the cost rate on the horizontal axis, and indicates the approximate line (dotted line) calculated by MS Excel.

From Table 1, Jarque-Beta test (JB) is obtained as follows [26].

$$\begin{aligned} JB &= \frac{n}{6}\left[S^2 + \frac{1}{4}(K-3)^2\right] \\ &= \frac{15}{6}\left[(-0.805)^2 + \frac{1}{4}(-0.482-3)^2\right] = 9.169 \cong 9 \end{aligned} \tag{40}$$

where, n is the number of observations, S is the sample skewness, and K is the sample kurtosis.

Regarding with a skewness and kurtosis, if JB is smaller than nine, the sample data obey a normal distribution. JB test result has over 9 slightly. However, the value is regarded as almost nine. Therefore, the sample data obey a normal distribution in Figure 12.

10. **Conclusions.** Regardless of the size of a supply chain system, a mathematical model defined by Langevin-type stochastic differential equations is known to identify self-similarity. In this study, we applied the mathematical model to the Langevin equation for a supply chain system. The self-similarities of the supply chain were observed in the numerical simulation. We also assume that the throughput deviation exhibits a normal distribution. Moreover, we clarified that a supply chain system has f^{-1} fluctuations when the time constant distribution of the time correlation function is a uniform Poisson

Table 2. Cost table of six workers at Test-run 1 (asynchronous)

	WC	S1	S2	S3	S4	S5	S6
K1	60	80	80	100	80	80	80
K2	80	88	84	88	84	76	80
K3	40	80	104	100	88	88	104
K4	120	102	90	114	108	96	90
K5	90	90	120	108	96	90	90
K6	90	90	90	90	90	90	90
K7	120	160	160	240	160	168	160
K8	160	232	264	240	232	256	264
K9	120	112	112	120	112	112	112
Total	880	1034	1104	1200	1050	1056	1088

Table 3. Cost table of six workers at Test-run 2 (synchronous 1)

	WC	S1	S2	S3	S4	S5	S6
K1	80	80	96	80	80	80	80
K2	80	80	80	80	80	80	80
K3	80	80	80	80	80	80	80
K4	120	150	150	120	120	120	120
K5	120	120	120	120	120	120	120
K6	120	120	120	120	120	120	120
K7	160	160	160	160	160	160	160
K8	160	216	216	176	184	160	160
K9	160	160	160	160	160	160	160
Total	1080	1166	1182	1096	1104	1088	1080

Table 4. Cost table of six workers at Test-run 3 (synchronous 2)

	WC	S1	S2	S3	S4	S5	S6
K1	80	72	76	72	80	80	80
K2	80	72	72	72	80	80	80
K3	80	84	84	84	80	80	80
K4	120	78	66	66	120	120	120
K5	120	96	96	102	120	120	120
K6	120	108	108	108	120	120	120
K7	160	112	112	104	160	160	160
K8	160	176	176	160	160	160	160
K9	160	200	200	200	160	160	160
Total	1080	998	990	968	1080	1080	1080

Table 5. Cost table of six workers at Test-run 4 (synchronous 2)

	WC	S1	S2	S3	S4	S5	S6
K1	80	72	76	72	72	72	72
K2	80	72	72	72	72	72	72
K3	80	84	84	84	84	84	84
K4	96	78	66	66	78	78	78
K5	96	96	96	102	120	96	96
K6	96	108	108	108	108	108	108
K7	160	112	112	104	112	112	104
K8	160	176	176	176	176	176	176
K9	160	200	200	200	200	200	200
Total	1008	998	990	984	1004	998	990

distribution and has f^{-2} fluctuations when the time constant distribution of the time correlation function is a uniform distribution.

With respect to potential energy, if a supply chain has potential energy, this potential energy needs to be defined. Determining this definition of potential energy will be the focus of future study.

In the analysis results of the cost rate data based on the lead time, we clarified that the probability distribution exhibits a normal distribution. Furthermore, we clarified that the probability distribution for the absolute value of the cost rate deviation based on the lead time exhibits a power-law distribution.

We have demonstrated that self-similarity exists in the supply chain system. Regarding the management strategy that leads to excessive production or to a state of excessive order entries, we propose that it is possible to increase profit by adopting an ideal strategy.

In this study, the normal distribution of the cost rate deviation based on the lead time is assumption.

REFERENCES

[1] R. N. Manteqna and H. E. Stanley, Scaling behavior in the dynamics of an economic index, *Nature*, vol.376, no.6, pp.46-49, 1995.

[2] M. Takayasu, K. Fukuda and H. Takayasu, Application of statistical physics to the Internet traffics, *Physica*, vol.274, pp.140-148, 1999.

[3] M. Takayasu, H. Takayasu and K. Fukuda, Dynamic phase transition observed in the Internet traffic flow, *Physica*, vol.277, pp.245-255, 2000.

[4] M. Takayasu, H. Takayasu and T. Sato, Critical behavior and 1/f noise in information traffic, *Physica*, vol.233, pp.824-834, 1996.
[5] H. Takayasu, Statistical physics style approach of price variation, *Physical Society of Japan Journal*, vol.54, no.1, pp.3-10, 1999.
[6] H. Takayasu, Markets price fluctuation seen as critical fluctuation, *Japan Society for Industrial and Applied Mathematics Journal*, vol.11, no.4, pp.56-65, 2001.
[7] K. Shirai, Y. Amano, S. Omatu and E. Chikayama, Power-law distribution of rate-of-return deviation and evaluation of cash flow in a control equipment manufacturing company, *International Journal of Innovative Computing, Information and Control*, vol.9, no.3, pp.1095-1112, 2013.
[8] K. Shirai and Y. Amano, Production density diffusion equation propagation and production, *IEEJ Transactions on Electronics, Information and Systems*, vol.132-C, no.6, pp.983-990, 2012.
[9] K. Shirai and Y. Amano, A study on mathematical analysis of manufacturing lead time – Application for deadline scheduling in manufacturing system, *IEEJ Transactions on Electronics, Information and Systems*, vol.132-C, no.12, pp.1973-1981, 2012.
[10] K. Shirai, Y. Amano and S. Omatu, Process throughput analysis for manufacturing process under incomplete information based on physical approach, *International Journal of Innovative Computing, Information and Control*, vol.9, no.11, pp.4431-4446, 2013.
[11] K. Shirai, Y. Amano and S. Omatu, Propagation of working-time delay in production, *International Journal of Innovative Computing, Information and Control*, 2013.
[12] K. Shirai, Y. Amano and S. Omatu, Consideration of phase transition mechanisms during production in manufacturing processes, *International Journal of Innovative Computing, Information and Control*, vol.9, no.9, pp.3611-3626, 2013.
[13] K. Shirai, Y. Amano and S. Omatu, Improving throughput by considering the production process, *International Journal of Innovative Computing, Information and Control*, vol.9, no.12, pp.4917-1930, 2013.
[14] K. Nishioka, Y. Mizutani, H. Ueno et al., Toward the integrated optimization of steel plate production process – A proposal for production control by multi-scale hierarchical modeling –, *Synthesiology*, vol.5, no.2, pp.98-112, 2012.
[15] L. Sun, X. Hu, Y. Fang and M. Huang, Knowledge representation for distribution problem in urban distribution systems, *International Journal of Innovative Computing, Information and Control*, vol.6, no.9, pp.4145-4156, 2010.
[16] L. Hu, D. Yue and J. Li, Availability analysis and design optimization for repairable series-parallel system with failure dependencies, *International Journal of Innovative Computing, Information and Control*, vol.8, no.10(A), pp.6693-6705, 2012.
[17] S. Hiiragi, The significance of shortening lead time from a business perspective, *MMRC*, University of Tokyo, http://merc.e.u-tokyo.ac.jp/mmrc/dp/index.html, 2012 (in Japanese).
[18] N. Ueno, M. Kawasaki, H. Okuhira and T. Kataoka, Mass customization production planning system for multi-process, *Journal of the Faculty of Management and Information Systems, Prefectural University of Hiroshima*, no.1, pp.183-192, 2009 (in Japanese).
[19] H. Tasaki, *Thermodynamics – From a Modern Point of View (New Physics Series)*, Baifukan, Co., LTD., 2000.
[20] H. Fujisaka, K. Fukushima, M. Inoue and T. Yamada, Phase transition in chaos and on-off intermittency, *The Physical Society of Japan*, vol.51, no.11, pp.813-820, 1996.
[21] R. Yamamoto, T. Nakaturu, K. Miyajima and M. Ishikawa, On the mathematical modeling of order-disorder transition by stochastic partial differential equations, *Yamaguchi University Faculty of Engineering Report*, vol.50, no.1, pp.45-51, 2012.
[22] S. Satoh and K. Fujimoto, Passivity based control of stochastic port-hamiltonian systems, *Journal of the Society of Instrument and Control Engineers*, vol.43, no.3, pp.1-8, 2006.
[23] T. C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.
[24] K. Kitahara, *Nonequilibrium Statistical Mechanics*, Iwanami Co., LTD., 2000.
[25] T. Imamura, *Mathematics in Probability Field*, Iwanami Co., LTD.
[26] M. J. Carlos and K. B. Anil, Efficient tests for normality, homoscedasticity and serial independence of regression residuals: Monte Carlo evidence, *Economics Letters*, vol.7, no.4, pp.313-318, 1981.

Appendix A. **Conducting process of Langevin equation.** From Equation (20), we obtain the following:

$$\frac{dh_1(t)}{dt} = \left[\frac{\Omega_E}{\rho}\right]^{-1}\frac{dh_0}{dt} = \left[\frac{\Omega_E}{\rho}\right]^{-1}[-rh_0(t) + f_0(t)] \tag{41}$$

We generalize Equations (16) and (17), and we obtain as follows:

$$qh_{n-1}\left(\frac{\rho}{\Omega_E}\right) = h_n(t) \tag{42}$$

$$pf_{n-1}\left(\frac{\rho}{\Omega_E}\right) = f_n(t) \tag{43}$$

Then we obtain Equation (44) after rewriting Equations (42) and (43).

$$\begin{aligned}\frac{dh_1(t)}{dt} &= \left[\frac{\Omega_E}{\rho}\right]^{-1}[-rq^{-1}h_1(t) + f_0(t)] = \left[\frac{\Omega_E}{\rho}\right]^{-1}[-rq^{-1}h_1(t)] \\ &= \left[\frac{\Omega_E}{\rho}\right]^{-1}p^{-1}f_1(t) = -\left[\frac{\rho}{\Omega_E}\right]rh_1(t) + \left[\frac{\rho}{\Omega_E}\right]q\cdot p^{-1}f_1(t)\end{aligned} \tag{44}$$

Here, we assume the following:

$$\left[\frac{\rho}{\Omega_E}\right]\cdot q\cdot p^{-1} \equiv 1$$

From Equation (44), we obtain as follows:

$$\frac{dh_1(t)}{dt} + \left[\frac{\rho}{\Omega_E}\right]rh_1(t) = f_1(t) \tag{45}$$

where q and p satisfies Equation (46).

$$\left[\frac{\rho}{\Omega_E}\right]qp^{-1} = 1 \tag{46}$$

We replace Equation (46) to Equation (47) for example.

$$q = \sqrt{\frac{\Omega\Omega_E}{\rho}},\ \ p = \sqrt{\rho\frac{\Omega}{\Omega_E}},\ \ \forall\left(p^{-1} = \sqrt{\frac{\Omega_E}{\rho\Omega}}\right) \tag{47}$$

Then q represents a diffusion coefficient.

$$\frac{\partial S}{\partial t} = D^2\frac{\partial^2 S}{\partial x^2},\ \ D^2 = \frac{\kappa}{\rho c}\ \ \left(D = \sqrt{\frac{\kappa}{\rho c}}\right) \tag{48}$$

Appendix B. **f^{-1} Fluctuations of $h_n(t)$.** We substitute Equation (12) to Equation (9), and we obtain the following:

$$\begin{aligned}S_{h_n}(f_{req}) &= \int_0^\infty dt\cos(2\pi f_{req}(t))D_{req} < |h_n|^2 > \exp\left(-\frac{t}{\tau_n}\right) \\ &= D_{req} < |h_n|^2 > \int_0^\infty \exp\left(-\frac{t}{\tau_n}\right)\cos(2\pi f_{req}t)dt\end{aligned} \tag{49}$$

where if $(t/\tau_n) = \xi_i$, $dt = \tau_n d\xi_i$. Then, we obtain Equation (50) from Equation (49).

$$\begin{aligned}
S_{T_i}(f_{req}) &= D_{req} < |h_n|^2 > \int_0^\infty \exp(-\xi_i)\cos(2\pi f_{req}\xi_i\tau_n)d\xi_i \\
&= D_{req}\tau_n < |h_n|^2 > \\
&\quad \times \Big[\frac{-\cos(2\pi f_{req}\tau_n\xi_i) + 2\pi f_{req}\tau_n \sin(2\pi f_{req}\tau_n\xi_i)}{1+(2\pi f_{req}\tau_n)^2} \times \exp(-\xi_i)\Big]_0^\infty \\
&= D_{req} < |h_n|^2 > \frac{\tau_n}{1+(2\pi f_{req}\tau_n)^2} \qquad (50)
\end{aligned}$$

Therefore, we can obtain Equation (13).
Here, if a time constant τ_n is proportional to τ_n^{-1}, we can obtain the following:

$$\begin{aligned}
< S_{T_i}(f_{req}) > &= \int_0^\infty \frac{d\tau_n}{\tau_n} \cdot \frac{D_{req}\cdot\tau_n}{(2\pi f_{req}\tau_n)^2+1} < |h_n|^2 > \\
&= D_{req}\int_0^\infty d\xi_i \frac{< |h_n|^2 >}{(2\pi\xi_i)^2+1} \cdot \frac{1}{f_{req}}, \quad {}^\forall f_{req}\tau_n = \xi_i \qquad (51)
\end{aligned}$$

We can also obtain Equation (52) after calculating of Equation (51).

$$\begin{aligned}
< S_{T_i}(f_{req}) > &= D_{req} < |h_n|^2 > \int_0^\infty d\xi_i \frac{1}{(2\pi\xi_i)^2+1} \cdot \frac{1}{f_{req}} \\
&= D_{req} < |h_n|^2 > \int_0^\infty \Big(\frac{1}{2\pi}\Big)^2 \cdot \frac{1}{(\xi_i)^2 + (\frac{1}{2\pi})^2} \cdot \frac{1}{f_{req}} \\
&= D_{req} < |h_n|^2 > \Big(\frac{1}{2\pi}\Big)\Big\{\arctan(2\pi\xi_i)\Big\} \cdot \frac{1}{f_{req}} \qquad (52)
\end{aligned}$$