International Journal of Innovative Computing, Information and Control
Volume **12**, Number **5**, October **2016**
ICIC International ©2016 ISSN 1349-4198
pp. 1615–1628

ANALYSIS OF FLUCTUATIONS IN PRODUCTION PROCESSES USING BURGERS EQUATION

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Received February 2016; revised June 2016

Abstract. *We report an analysis of the fluctuations in the lead time of production processes by applying Burgers equation of fluid dynamics, which constrains the state variables in an internal process. The propagation of fluctuations corresponds to a fluctuation in the lead time. The factors of such fluctuations include an uncertainty of logistics, uncertainty of production planning, and stochastic characteristics of the start (order) time series. We obtained the actual fluctuation data indicated by the starting time series (order time-series) of the lead time period in a batch production system. The throughput of production processes such as the fluctuation in a turbulence spot is affected by volatilities in the same manner as the coefficient of diffusion equation. Based on the fact that the starting time series of the lead time period in production processes has intermittent on-off characteristics, we confirm that reducing volatilities was the most important factor for improving production processes. The production efficiency of synchronous processes became clear from the actual data. For further verification, we confirmed the benefit of using the synchronization process by performing a dynamic simulation.*
Keywords: Throughput deviation, Burgers equation, Fluctuation, Lead time, Production process

1. **Introduction.** Many currently implemented production systems are mechanized and highly integrated with information technologies, which creates systems where human intervention is unnecessary. In certain aspects of the production system, there is a high volume of build-to-order manufacturing that requires human intervention in the production process [1, 3]. In small- and medium-sized enterprises, human intervention constitutes a significant part of the production process, and revenue can sometimes be greatly affected by human behavior. Therefore, with respect to human intervention with outside companies, a deep analysis of the production process and human collaboration is necessary to understand the potential negative effects of human intervention [1, 3]. Naturally, the effect of human behavior is not just a problem with small- and medium-sized companies; it must be regarded as one of the major problems that may occur when humans directly intervene in the production process [4, 5, 10].

In general, the potential uncertainties should be considered before proceeding with a system that combines human intervention (Internal force) with outside companies (External force) in the production system [6, 7]. With respect to two elements in a production system, a total system is formed by connecting the two elements. In this case, a system with certain uncertainties will be formed when connecting "human intervention" and "outside companies" in a production system. In general, an important concept in the production system is to develop the best system that results in efficient production. However, in most analyses of the production process, researchers have not taken advantage of the noise inherent in the system. Such noise may have a unique usefulness in the system.

Thus, we have been researching mathematical modeling and system evaluations from a physical point of view to develop "mathematical production engineering" in order to develop a mathematical system for describing production processes. In a previous study of stochastic modeling, we considered the internal force and external force as parameters in a production system. The correlation of lead time vs. throughput is important for implementing the overall synchronization as a strategy. We had reported a production system with an intervention of workers in the prior study [6, 7]. In case of a production flow system with human intervention, we need to fulfill an empirical analysis of worker-specific production ability. Thus, to achieve optimal general production systems, knowledge of the importance of biological fluctuations in the system is important.

In our previous study, an on-off intermittency exists in the rate-of-return and lead time deviations of production processes. In physics, an on-off intermittency is presented in case of power-law distributions, phase transitions, and self-similar phenomena. In the production process described in this study, we observed on-off intermittency on a lead time data with respect to time series outset [8]. Previously, we have reported that by creating a state in which the production density of each process corresponds to physical propagation, the manufacturing process is most appropriately described using a diffusion equation [1]. In other words, if the potential of the production field (stochastic field) is minimized, the equation is defined by the production density function $S_i(x,t)$ and the constraint is described using an advective diffusion equation to determine the transportation speed ρ [1, 18].

To enable efficient application to a production system, we have proposed a mathematical model that focuses on the selection process and production lead time adaptation mechanism. To model the throughput time for a production demand/manufacturing system in the manufacturing stage, the dynamic behavior is derived using a lognormal stochastic differential equation. Using this model, the evaluation equation for the compatibility condition production lead time is defined using the risk-neutral integral, and the evaluation formula for the above conditions is calculated. Furthermore, by performing the synchronization process, the throughput for the manufacturing process is reduced [3].

In this study, we utilize Burgers equation for analyzing the fluctuations in the lead time of production processes. The factors causing fluctuations include the following:

- Uncertainty of logistics
- Uncertainty of production planning
- Stochastic characteristics of the order and start time series

Our research findings are as follows.

- The fluctuation in the lead time is caused by the propagation of the fluctuation of the state variables constrained by Burgers equation of fluid dynamics.
- Based on our current results, we can observe and link the on-off intermittence in time with the fluctuations that we previously reported in 2014 [8].
- A phenomenon similar to the occurrence in turbulent flow fields discussed in fluid dynamics is observed in production processes.
- We derive the Burgers equation by recognizing the graph of the start time series (order time-series) of the lead time period in production processes.
- The diffusion coefficient affects the fluctuation in turbulent spots in fluid mechanics.

- When the configuration parameters of the diffusion coefficient are considered as a trend coefficient and volatility, a production process can approach a synchronous process such as laminar flow in fluid dynamics by reducing the volatility of the production processes.
- Based on our actual data, we show the reductions in volatility, which led to an improved production throughput.
- We also implement a dynamic simulation to evaluate and confirm the effectiveness of synchronous and asynchronous processes.

To the best of our knowledge, this is the first study on the factor of fluctuations in production processes.

2. Fluctuation Analysis by Burgers Equation.

2.1. Propagation of production density. Figure 1 shows that connection between processes can be treated as diffusive propagation of products [1]. In Figure 1, u and n represent the throughput and production density, respectively [1]. In fluid dynamics, S represents the cross-sectional area; the number density continuity equation is described as follows:

$$\Delta(nS\Delta x) = n(t,x)u(t,x)\Delta tS - n(t,x+\Delta x)u(t,x+\Delta x)\Delta t \tag{1}$$

$$\left(\frac{\Delta n}{\Delta t}\right)_x = -\frac{n(t,x+\Delta x)u(t,x+\Delta x) - n(t,x)u(t,x)}{\Delta x} \tag{2}$$

$$\frac{\partial n}{\partial t} = -\frac{\partial(nu)}{\partial x} \tag{3}$$

The left-hand side of the second term $u\frac{\partial u}{\partial t}$ is the advection term. Now, letting $u = c$ (constant value), we consider the following equation.

$$\frac{\partial}{\partial t}u + c\frac{\partial}{\partial x}u = 0 \tag{4}$$

Equation (26) denotes a linear wave motion traveling to $+x$ direction at a constant speed c. Then, In Figure 2, when the advection speed changes, Figure 2(a) shows that the

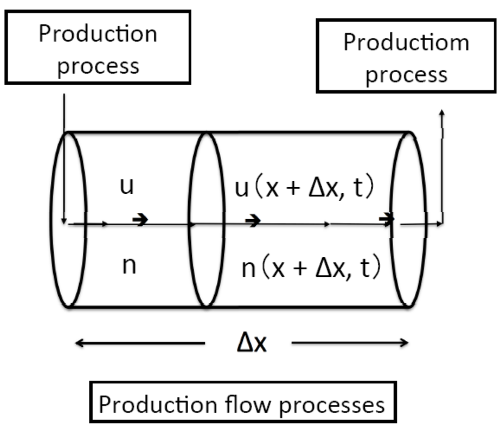

FIGURE 1. Production flow processes

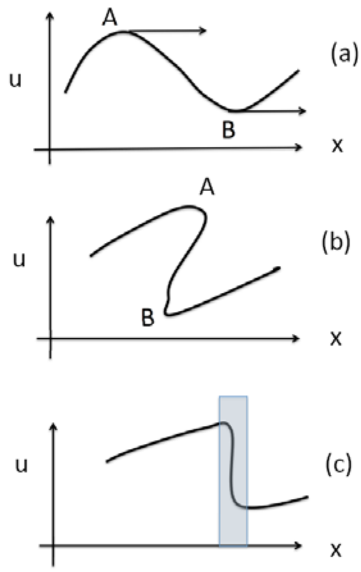

FIGURE 2. Bottleneck phenomenon similar as waves

A part moves quickly to the right, and the distance between AB is shortened gradually because the B part moves slowly. Figure 2(b) shows that the A part catches up with the B part and overtakes it after a certain time has elapsed, following which the wave collapses. Figure 2(c) shows that the dissipation area suppresses processes like the wave until a limited gradient form when the spatial gradient becomes sharp. The fill area also shows an area where dissipation occurs [16]. Figure 3 depicts a production process that is termed as a production flow process. This production process is employed in the production of control equipment. In this example, the production flow process consists of six stages. In each step S1-S6 of the manufacturing process, materials are being produced. Equation (3) is a continuous equation describing the throughput. The bottleneck occurs at some stage of the process in Figure 3.

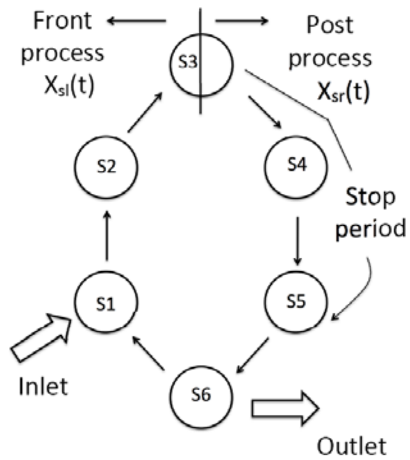

FIGURE 3. Bottleneck period in production flow processes

2.2. **Mathematical modeling by Burgers equation.** We consider the fluctuation characteristics of the turbulent and production lead time of production field by using the Burgers equation. The factors causing fluctuations include the following again:

- Uncertainty of logistics
- Uncertainty of production planning
- Stochastic characteristics of the order and start time series

Linkage of these factors cause the fluctuation; that is, we reported that an on-off intermittency was observed, and then a bottleneck occurs in the production processes.

Figure 4 shows a boundary surface of fluctuation characteristics. In this study, we used the boundary surface characteristics of the fluctuations to develop a solution for Burgers equation.

Then, the corresponding Burgers equation that ignores the pressure term is as follows [2]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} \tag{5}$$

By executing Cole-Hopf transformation [9],

$$u = -\nu\frac{\partial}{\partial x}\ln\psi \tag{6}$$

We obtain as

$$\frac{\partial \psi}{\partial t} = D_\nu\frac{\partial^2 \psi}{\partial x^2} \tag{7}$$

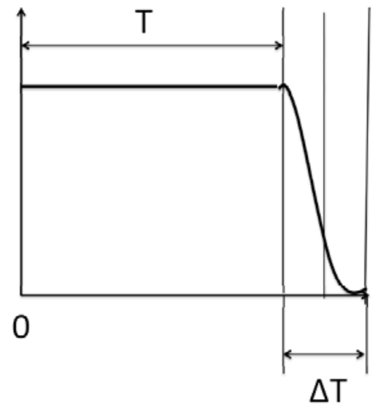

FIGURE 4. Boundary surface of fluctuation characteristics

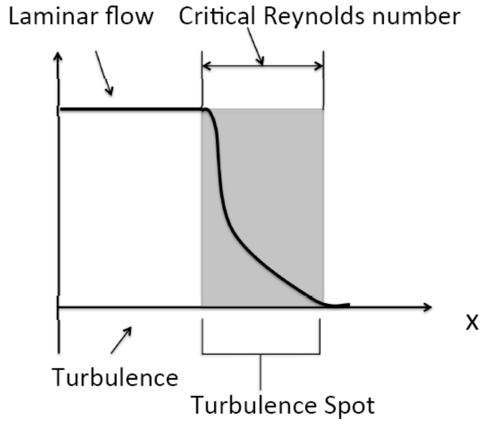

FIGURE 5. Critical Reynolds number (turbulence spot)

Here, we considered the model of production processes in detail by the above described model.

Definition 2.1. *$C(t,x)$ is a production density. v_r is production speed.*

A production flow is

$$\frac{\partial C}{\partial t} + \frac{\partial J}{\partial x} \tag{8}$$

Then,

$$J = Cv_r - D\frac{\partial C}{\partial x} \tag{9}$$

where D is a diffusion coefficient.
Then D is

$$D = \tau v_r^2 \tag{10}$$

where v_r is a convection time and τ is an average parts combination work time.
Further, v_r is

$$v_r = v_0\left(1 - \frac{C}{C_s}\right) \tag{11}$$

where production speed is assumed to depend on the production density.
From above results,

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)C - 2\left(\frac{v_0}{C_s}\right)C\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} = 0 \tag{12}$$

Then, we introduce the variable ξ for transformation of Equation (12).

$$\xi = -x + v_0 t \tag{13}$$

From Equation (13), Equation (12) is

$$\frac{\partial C}{\partial t} + aC\frac{\partial C}{\partial \xi} = D\frac{\partial^2 C}{\partial \xi^2} \tag{14}$$

where $a = (2v_0/C_s)$.
Equation (14) represents Burgers equation.

Here, we transform Equation (14).

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} = D_R\frac{\partial^2 C}{\partial x^2} \tag{15}$$

where

$$D_R = \left[\frac{(v_0/v_r)t_0}{\tau}\right]^{-1} \tag{16}$$

where t_0 is an average convection time of production density, and $x_0 = v_0 t_0$.

We execute Cole-Hopf transformation to Equation (15).

$$C = -D_h\frac{\partial}{\partial x}\ln h \tag{17}$$

$$\frac{\partial h}{\partial t} = D_h\frac{\partial^2 h}{\partial x^2} \tag{18}$$

Equation (19) represents a one-dimensional diffusion equation. We obtained the Burgers equation for production processes that are similar to turbulence models in fluid dynamics.

From Equation (15), we can obtain a particular solution in case of the following equation:

$$2\frac{\partial \varphi}{\partial t} + \varphi\frac{\partial \varphi}{\partial x} = D\frac{\partial^2 \varphi}{\partial x^2} \tag{19}$$

where $\varphi \leq 1$.

$$\varphi(t,x) = \frac{A}{2}\left[1 - \tanh\left\{\left(\frac{A}{4D}\left(x - \frac{A}{4}t\right)\right)\right\}\right] \tag{20}$$

where

$$\varphi = A \text{ as } \lim_{x\to+\infty}\varphi, \quad \varphi = 0 \text{ as } \lim_{x\to-\infty}\varphi \tag{21}$$

We can obtain a traveling wave as $speed = A/4$ and $Width = (16D)/A^2$. The speed and width approach similar values of $A/4$ and $A/(4D)$, respectively, in Equation (21).

Based on Figure 5, the transition between laminar flow and turbulent flow occurs in production processes when an improvement or change of the endogenous parameters is made. A proper understanding of the critical value of the Reynolds number in the vicinity of the turbulence spot is required. This value needs to be defined for each production process; hence, formulating a mathematical model as its foundation is of utmost importance. The turbulence spot represents a fluctuation in free energy. Therefore, a synchronous status can be approached if the turbulence has a reduced spot width and the management person confines the possible production flow to a narrow region between laminar and turbulent flow. Therefore, when $\varphi(t,x)$ is considered as the continuum approximation of the variable throughput deviation between processes, in general, we consider the Burgers model as follows:

$$\xi\frac{\partial \varphi(t,x)}{\partial t} + \varphi(t,x)\frac{\partial \varphi(t,x)}{\partial x} = D\frac{\partial^2 \varphi(t,x)}{\partial x^2} \tag{22}$$

where $\varphi(t,x) \leq 1$.

The condition of the variable's x follows Equation (21). In other words, $x \to \infty$ represents an extremely large value that approaches bottleneck synchronization, and $x \to 0$ ignores processes.

$$\varphi(t,x) = \frac{A}{\xi}\left[1 - \tanh\left\{\left(\frac{A}{2\xi D}\left(x - \frac{B}{2\xi}t\right)\right)\right\}\right] \tag{23}$$

where the value of no time change $\varphi(t,x)$ represents $\varphi_0(x)$ $(B=0)$.

$$\varphi_0(t,x) = \frac{A}{\xi}\left(1 - \tanh\frac{1}{2}\left(\frac{A}{\xi}\right)D^{-1}\right) \tag{24}$$

where let $-\eta \le x \le +\eta(\eta > 0)$. x represents a deviation.

2.3. **Fluctuation function using Ginzburg-Landau (GL) free energy.** GL is defined as follows.

Definition 2.2.

$$F(\varphi) = \int_\Omega \left[\frac{k^2}{2}|V\varphi|^2 + f(\varphi)\right] dV, \quad V \in \Omega \tag{25}$$

As gradient system,

$$\tau\frac{\partial\varphi}{\partial t} = -\frac{\delta F(\varphi)}{\delta\varphi} = k^2V^2\varphi - f'(\varphi) \tag{26}$$

When the GL energy changes (i.e., fluctuates), the common understanding is that the fluctuations occur only near the transition point. Thus, when there is no time change,

$$\int_\Omega k^2\left[\frac{\partial^2\varphi_0}{\partial x^2}\right]\left(\frac{\partial\varphi_0}{\partial x}\right)dx - \int_\Omega\left[\frac{\partial\varphi}{\partial\varphi_0}\right]\left(\frac{\partial\varphi_0}{\partial x}\right)dx \tag{27}$$

By calculation of Equation (27) [17], we obtain

$$F(\varphi_0) = k\int_0^1 \sqrt{2f(\varphi_0)}\,d\varphi_0 \tag{28}$$

$F(\varphi)$ in Equation (28) represents the energy density of a critical vicinity of fluctuation in the production flow. Thus, $F(\varphi)$ is determined by the system parameters and its potential shape. From the normalized lead time of Figures 6 and 7, the normalized lead time of deviation, which governs the progression of processes, generates the phenomenon of on-off intermittency [8].

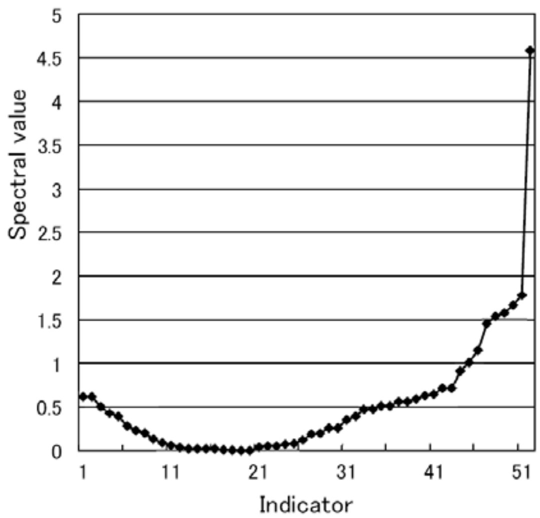

FIGURE 6. Fluctuation spectrum of the batch production normalized lead time

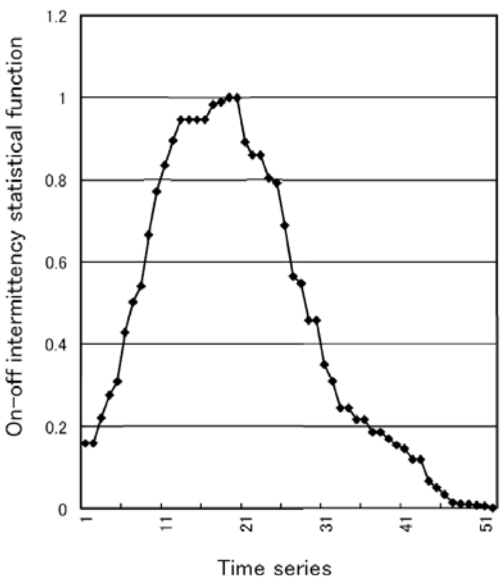

FIGURE 7. On-off intermittency statistical function $\exp(-t\cdot S(\Phi))$ on the batch production normalized lead time

Next, we propose a mathematical model with a throughput improvement plan.

$$\frac{\partial Q}{\partial t} + \rho Q \frac{\partial Q}{\partial x} = a\left[\frac{V(\Delta\varphi) - Q}{\tau}\right] + D\frac{\partial^2 Q}{\partial x^2} + Z(t,x) \tag{29}$$

where $Q \equiv Q(t,x)$ represents a throughput function, $V(\Delta\varphi)$ is determined by a lead time deviation, and $Z(t,x)$ is the distributed white noise. Equation (29) is the Burgers equation with distributed white noise, and systems derived by Equation (29) occur in an on-off intermittency [8].

The normalized lead times of Figures 14-17 show the snap at the time of the start of the process lead time (before/after reclassification). Therefore, $\varphi(t)$ is equal to a time progress rate and thus φ is a function for only time dependence.

Assumption 2.1.

$$\frac{\partial \varphi(t)}{\partial t} = \alpha\varphi(t), \quad 0 \le t \le T \tag{30}$$

Then, by adding the noise term $Z(t)$ to Equation (30),

$$\frac{\partial \varphi(t)}{\partial t} = (\alpha + Z(t))\varphi(t), \quad < Z(t) \cdot Z'(t) >= 2s\delta\left(t - t'\right) \tag{31}$$

Next, let a starting point be to variable x under a continuous process. We obtain

$$\begin{gathered}\frac{\partial \varphi(t,x)}{\partial t} = (\alpha + Z(t,x))\,\varphi(t,x) + D\frac{\partial^2 \varphi(t,x)}{\partial x^2}, \\ < Z(t,x) \cdot Z\left(t', x'\right) >= 2s\delta\left(t - t'\right)\delta\left(x - x'\right)\end{gathered} \tag{32}$$

According to a general analysis of statistical mechanics, which is Hopf-Cole transformation [9]:

$$\varphi(t,x) = \exp(h(t,x)) \tag{33}$$

Moreover,

$$U(t,x) = -\frac{\partial h(t,x)}{\partial x} \tag{34}$$

Then, we obtain after execution of Hopf-Cole transformation as follows:

$$\frac{\partial U(t,x)}{\partial t} + 2DU(t,x)\frac{\partial U(t,x)}{\partial x} = D\frac{\partial^2 U(t,x)}{\partial x^2} + \frac{\partial Z(t,x)}{\partial t} \tag{35}$$

3. **Actual Data Examples from a Production Process with a Nonlinearity.** We present actual data examples from both open and cyclic production flow process with nonlinearities. With respect to the actual data in cyclic production processes, in Table 1, test run3 indicates a best value for the throughput in the three types of theoretical working time. test run2 is an ideal production method. However, because it is difficult for talented worker, test run3 is a realistic method. Please see the actual data to [4, 11].

Table 1. Correspondence between the table labels and the test-run number

	Production process	Working time	Volatility
test run1	Asynchronous process	627(min)	0.29
test run2	Synchronous process	500(min)	0.06
test run3	"Synchronization with preprocess" method	470(min)	0.03

3.1. **Example of the cyclic production flow process.** Figure 8 depicts a production process that is termed as a production flow process. This production process is employed in the production of control equipment. In this example, the production flow process consists of six stages. In each step S1-S6 of the production process, equipments are being produced.

The direction of the arrows represents the direction of the production flow. In this process, production materials are supplied through the inlet and the end-product is shipped from the outlet. For this flow production system, we make the following two assumptions.

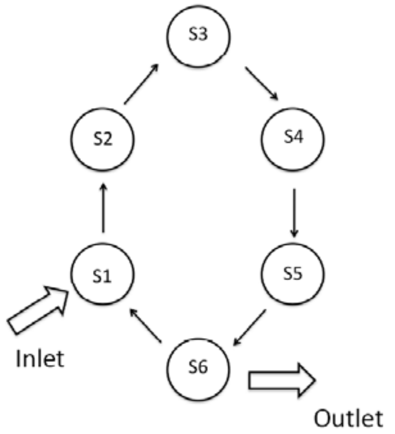

FIGURE 8. Cyclic production flow process

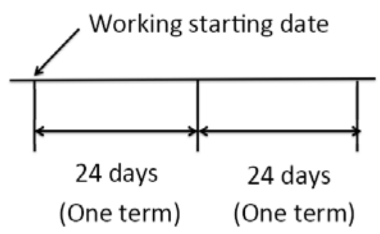

FIGURE 9. Actual work across the two periods

3.2. **Example of the open production flow process.** After we observed the nonlinear characteristics in the production processes, we attempted to improve the throughput [3]. At present, we have maintained a synchronized process. Using the asynchronous logistics phenomenon and supply chain delays, we present a throughput improvement example, in which a production flow process is used for throughput improvement, as shown in Figure 3.

Here, we investigated improved and standard process flows using a control device as an example. As a result, we found that post-process priority is appropriate for improving the throughput. Using a buffer of the previous process to overcome bottlenecks in the post process, the previous process can synchronize the post process, leading to significantly improved lead times.

The actual manufacturing is across the two periods in Figure 9. In Figure 9, there are about 24 days which is the longest lead time in one period. In case of production in uncertainty situation, it is an important issue to decide the production capacity by any method.

3.3. **Verification by actual data.** The current business style is a complete make-to-order production system and the production process is a batch process. Figures 6 and 7 show the deviation of the normalized lead time of a batch production. C1, C2, D1, D2 in Figure 10 become C1', C2', D1', D2' by moving the work start time respectively. P1 in Figure 11 represents the movement of the working power to W3. P2 in Figure 11 represents the movement of the working power to W5.

Therefore, Figure 10 and Figure 11 show the production processes before and after on-off intermittency, respectively. Figures 12 to 14 show the normalized lead time data before reclassification of production processes and corresponding to Figure 10. Figures 15 to 17 show the normalized lead time data after reclassification of production processes

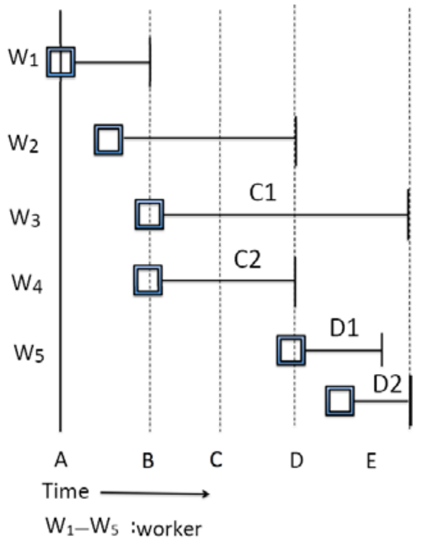

FIGURE 10. Process before managing of processes

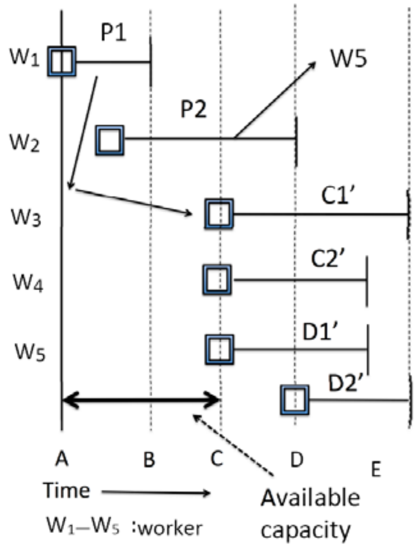

FIGURE 11. Process after managing of processes

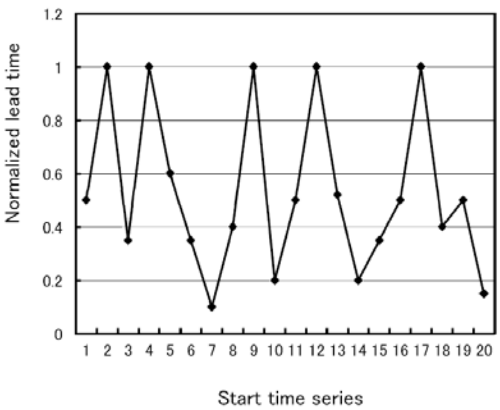

FIGURE 12. Normalized lead time data (before reclassification)

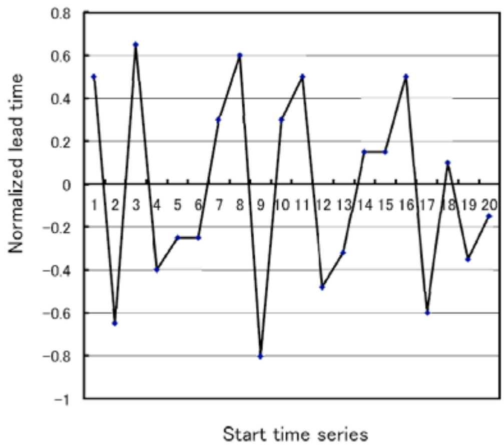

FIGURE 13. Normalized lead time data (before reclassification)

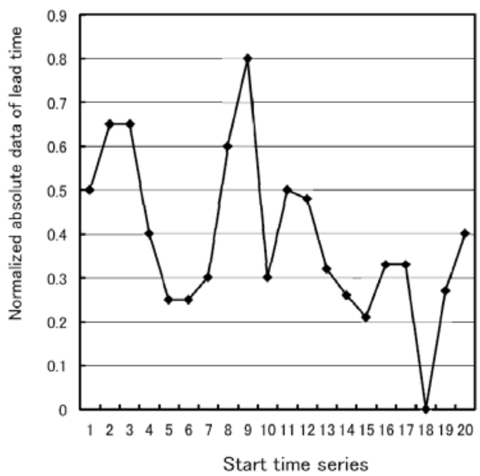

FIGURE 14. Normalized lead time data (before reclassification)

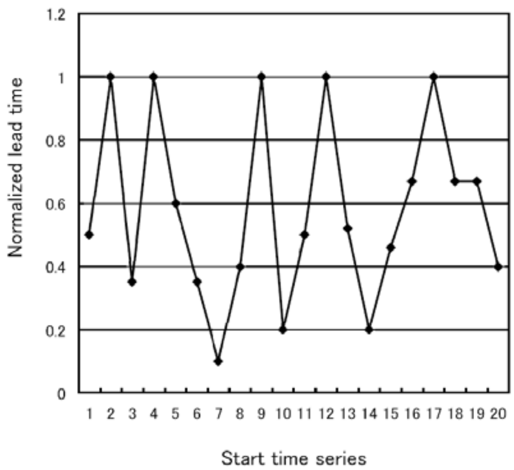

FIGURE 15. Normalized lead time data of deviation (after reclassification)

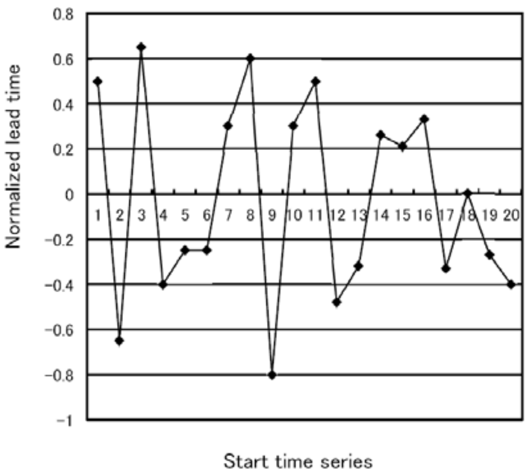

FIGURE 16. Normalized lead time data (after reclassification)

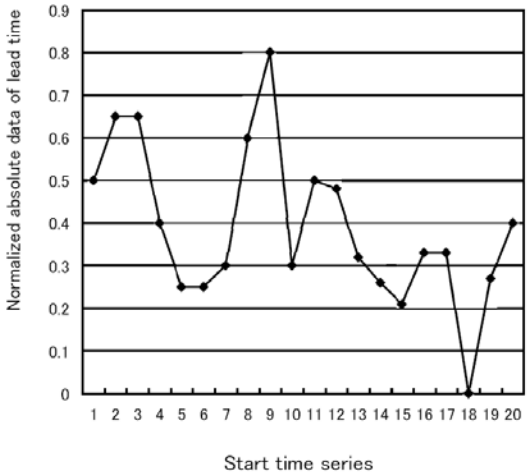

FIGURE 17. Normalized absolute data of lead time deviation data (after reclassification)

TABLE 2. Normalized lead time/volatility before reclassification of production process

Lead time	Volatility
9.67	0.236

TABLE 3. Normalized lead time/volatility after reclassification of production process

Lead time	Volatility
9.67	0.165

and and corresponding to Figure 11. Our strategy was to change the start time of the production and reduce the worker's volatility, as shown in Figure 11. After reconstructing the process shown in Figure 11, we could handle sudden orders by appropriately managing the available manpower and prevent opportunity loss. As a result, we increased monthly shipments.

From Table 2 and Table 3, no change was observed in the actual lead times. However, the volatility is reduced. In the process diagram, the time allocated for the operator is the first half of the entire process. We could offer a sudden customer support for the production. As a result, the production throughput also was improved.

Definition 3.1. *Throughput coefficient based on a standard production flow*

$$\eta \equiv \frac{[\textit{Number of production man-power}] \times [\textit{Number of real working time}]}{[\textit{Production risk rate}] \times [\textit{Reduction rate of lead time}]} \times \frac{1}{[\textit{Real working time of lead time}]} \tag{36}$$

If the numerator is constant, i.e., [production risk rate] = 1 and [real lead time] = constant, $\eta \cong 1.21$ (21% increase) in the improved production and $\eta \cong 1.35$ (35% increase) in the another improved production. Please see the reference in detail [12].

From the above description, by using a previous process as a buffer in a post process, we can realize synchronization between a previous process and post process. In other words, we have realized a post process with priority higher than the previous process.

3.4. **Dynamic simulation of production processes.** Regarding rate-of-return deviation, it was found that it conforms to log-normal distribution. From the analysis of mathematical models about rate-of-return deviation, we obtained the following conclusion. If an amount of money of order entries and an amount of money of production are stochastic, accumulated excessive order entries becomes of Brownian motion, and thus a random "fluctuation" occurs in hour to hour order entries and production even though it might be of a small degree. In addition, a rate of return is distributed log-normally, a cash flow of a target company proportional to a rate of return will be also distributed log-normally, naturally [15]. Therefore, we attempted to perform a dynamic simulation of the production process by utilizing the simulation system that NTT DATA Mathematical Systems Inc. (www.msi.co.jp) has developed. We conducted the simulation procedure in Figure 18. Please see the detail procedure in [11].

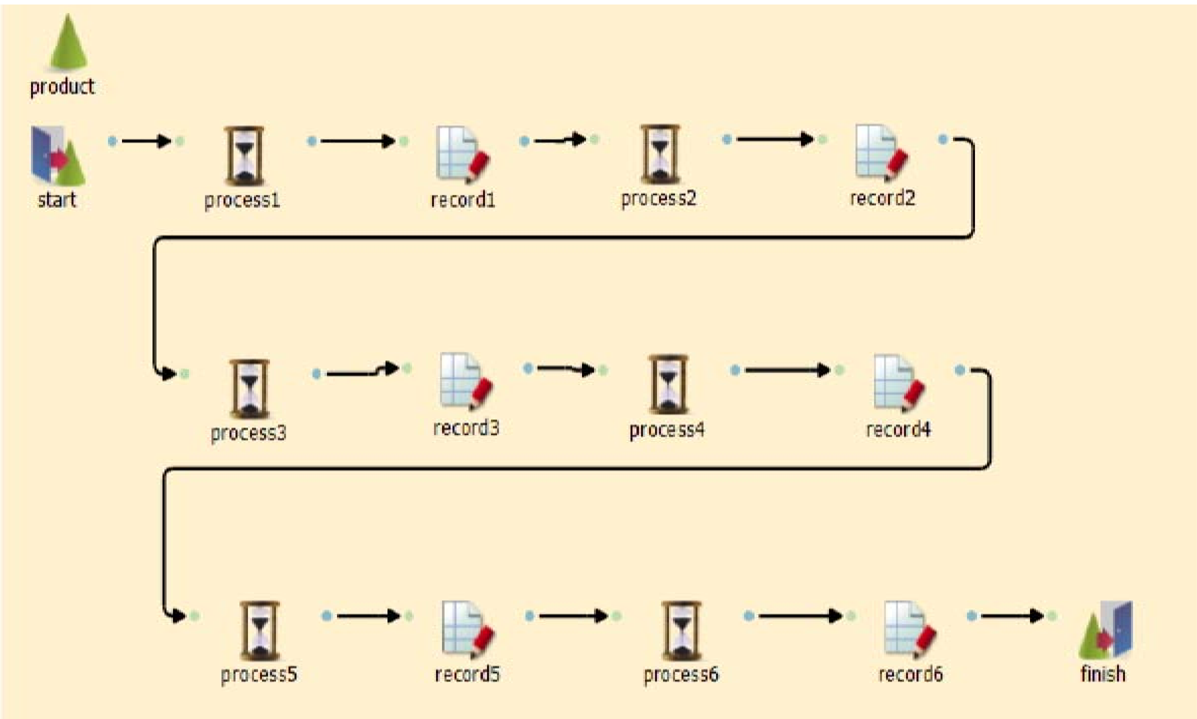

FIGURE 18. Simulation model of production flow system

With respect to the meaning of the individual parts in Figure 18, "record" calculates the worker's operating time, which is obtained by multiplying the specified WE data for the log-normally distributed random numbers in the data. Please see the reference about the data for Figure 19 [11].

Figure 19 shows the operating time of process 1-6 (record1-record6). As the working time of the synchronous process is less volatile, the work efficiency became higher than the asynchronous process. In Figure 19, the total working time of asynchronous and synchronous processes are 1241.7(sec) and 586.4(sec) respectively. The synchronous process shows more better production efficiency than the asynchronous process.

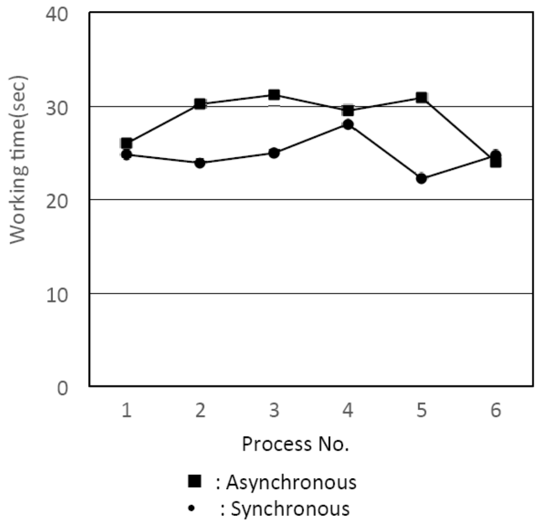

FIGURE 19. Working time for process number one through six

4. **Conclusion.** We clarified that the fluctuations in lead time were dependent on the state variable, which was a throughput deviation. The propagation of throughput deviation was restricted by Burgers equation of fluid dynamics. In our previous study, we reported that the normalized lead time data had an on-off intermittency. To verify our analysis, we represented actual data that were obtained before/after the managing of processes using the cyclic production flow process. Moreover, we presented results from a dynamic simulation to confirm the superiority of the synchronous process compared to the asynchronous process.

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